

LOOP CALCULATIONS IN QUANTUM MECHANICAL NON-LINEAR SIGMA MODELS WITH FERMIONS AND APPLICATIONS TO ANOMALIES¹

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ABSTRACT

We construct the path integral for one-dimensional non-linear sigma models, starting from a given Hamiltonian operator and states in a Hilbert space. By explicit evaluation of the discretized propagators and vertices we find the correct Feynman rules which differ from those often assumed. These rules, which we previously derived in bosonic systems [1], are now extended to fermionic systems. We then generalize the work of Alvarez-Gaumé and Witten [2] by developing a framework to compute anomalies of an n -dimensional quantum field theory by evaluating perturbatively a corresponding quantum mechanical path integral. Finally, we apply this formalism to various chiral and trace anomalies, and solve a series of technical problems: (i) the correct treatment of Majorana fermions in path integrals with coherent states (the methods of fermion doubling and fermion halving yield equivalent results when used in applications to anomalies), (ii) a complete path integral treatment of the ghost sector of chiral Yang-Mills anomalies, (iii) a complete path integral treatment of trace anomalies, (iv) the supersymmetric extension of the Van Vleck determinant, and (v) a derivation of the spin- $\frac{3}{2}$ Jacobian of Alvarez-Gaumé and Witten for Lorentz anomalies.

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1 Introduction

In this article we discuss Euclidean path integrals for one-dimensional systems with a Hamiltonian which is more general than $H(\hat{p}, \hat{x}) = T(\hat{p}) + V(\hat{x})$ ⁶, and their applications to anomalies. Namely we shall consider non-linear sigma models whose classical action is of the form⁷ $\int_{-\beta}^0 \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} dt$, plus fermionic extensions, which may be, but need not be, supersymmetric. This is the area of quantum mechanical path integrals in curved space, a difficult and controversial subject [3, 4].

Quantum mechanical path integrals are of importance as toy models for path integrals for field theories. Because the former are finite, no renormalization is necessary and subtle issues can better be studied. In addition, quantum mechanical path integrals are useful because some quantities of n -dimensional quantum field theories can be calculated in a much simpler way by using the corresponding one-dimensional path integrals. The prime example are anomalies, which we shall discuss in detail in the second part of this article. In the first part we give a careful derivation of quantum mechanical path integrals and the Feynman rules to which they give rise.

This article is an extension of a previous article [1] on bosonic quantum mechanical path integrals to the fermionic case and to applications to anomalies. In section 2.1 we briefly review the bosonic path integrals, but most emphasis in section 2 and beyond is on the fermionic case. At the end of this introduction we shall state which results are new, but we shall start with a general introduction in which we discuss bosonic and fermionic systems on equal footing.

The basic problem we solve is the following. Given a Hamiltonian $\hat{H}(\hat{p}, \hat{x}, \hat{\psi}^\dagger, \hat{\psi})$ with arbitrary but a priori fixed ordering of the bosonic operators \hat{p}_i, \hat{x}^i ($i = 1 \dots n$) and fermionic operators⁸ $\hat{\psi}^a, \hat{\psi}_a^\dagger$ ($a = 1 \dots n$), find a path integral representation for the transition element $\langle z, \bar{\eta} | \exp\left(-\frac{\beta}{\hbar} \hat{H}\right) | y, \chi \rangle$ where $|y, \chi\rangle$ is an eigenket of \hat{x}^i and $\hat{\psi}^a$ with eigenvalues y^i and χ^a , while $\langle z, \bar{\eta} |$ is an eigenbra of \hat{x}^i and $\hat{\psi}_a^\dagger$ with eigenvalues z^i and $\bar{\eta}_a$. Following Dirac [5] and Feynman [6] we shall insert $N - 1$ complete sets of x -eigenfunctions, N complete sets of p eigenstates and N complete sets of coherent states

$$\int |x\rangle \sqrt{g(x)} \langle x| d^n x = \int |p\rangle \langle p| d^n p = \int |\eta\rangle e^{-\bar{\eta}_a \eta^a} \langle \bar{\eta}| \prod_{a=1}^n (d\bar{\eta}_a d\eta^a) = \mathbf{1} \quad (1)$$

⁶Hats denote operators, but where no confusion arises we will omit them.

⁷Curved indices in space-time will be denoted by μ, ν, \dots . In the non-linear sigma model the corresponding indices will be denoted by i, j, \dots

⁸ For Majorana fermions, we shall formulate one approach where $a = 1, \dots, n$ and another where $a = 1, \dots, n/2$. We shall find it convenient to use fermionic operators satisfying anti-commutation relations without \hbar , $\{\psi^a, \psi_b^\dagger\} = \delta_b^a$, while $[\hat{x}^i, \hat{p}_j] = i\hbar \delta_j^i$ as usual.

to obtain a phase space path integral in the limit $N \rightarrow \infty$ of the form

$$\langle z, \bar{\eta} | \exp \left(-\frac{\beta}{\hbar} \hat{H} \right) | y, \chi \rangle \sim \int dx^i dp_i d\bar{\eta}_a d\eta^a e^{-\frac{1}{\hbar} \int_{-\beta}^0 L dt} \quad (2)$$

where $L = -ip_j \dot{x}^j + \hbar \bar{\eta}_a \dot{\eta}^a + H(p, x, \bar{\eta}, \eta)$. However several questions arise:

- (i) which is the relation between the operators $\hat{H}(\hat{p}, \hat{x}, \hat{\psi}^\dagger, \hat{\psi})$ and the functions $H(p, x, \bar{\eta}, \eta)$? Different operator orderings of \hat{H} must lead to different functions H . Are there particular orderings of \hat{H} for which H is equal to the naive classical Hamiltonian? After integrating out the momenta, is the action in the path integral invariant under the usual Einstein (general co-ordinate) transformations or supersymmetry transformations if the corresponding Hamiltonian \hat{H} commutes with the generators of these symmetries? (The answer is no).
- (ii) What are the Feynman rules needed to evaluate the path integral perturbatively?
- (iii) What is the precise meaning of the measure $dx^i dp_i d\bar{\eta}_a d\eta^a$? Is there a normalization constant in front of the path integral, or even factors $g(z)^\alpha g(y)^\beta$ where $g = \det(g_{ij})$? (The answer is yes)
- (iv) By adding couplings to external sources, one obtains propagators. These propagators are not the usual translationally invariant propagators because they must satisfy boundary conditions at $t = -\beta$ and $t = 0$. What boundary conditions for $p_i(t)$ and the fermions must be imposed?
- (v) When one is dealing with Majorana fermions, how should one define coherent states, and what is the Hilbert space in which \hat{H} is supposed to act. Should one impose boundary conditions both at $t = -\beta$ and $t = 0$ for Majorana fermions, and if so, how is this compatible with the fact that the equations of motion are linear in time derivatives?

These are some preliminary questions. Not all of them are new, but we shall automatically get answers by following our derivation of path integrals. These answers will be summarized in the conclusions. The most important question we solve has to do with Feynman rules. The propagators $\langle x^i(\sigma) x^j(\tau) \rangle$ and $\langle \psi^a(\sigma) \psi_b^\dagger(\tau) \rangle$ for configuration space path integrals (and in addition $\langle x^i(\sigma) p_j(\tau) \rangle$ and $\langle p_i(\sigma) p_j(\tau) \rangle$ for phase space path integrals) with $-1 \leq \sigma, \tau \leq 0$ contain factors $\theta(\sigma - \tau)$, where σ and τ are the time divided by β . In configuration space, there are contractions of $\dot{x}^i(\sigma)$

with $\dot{x}^j(\tau)$, which contain a factor $\delta(\sigma - \tau)$. When one computes Feynman graphs, one has to evaluate integrals over products of these distributions, for example

$$I = \int_{-1}^0 \int_{-1}^0 \delta(\sigma - \tau) \theta(\sigma - \tau) \theta(\tau - \sigma) d\sigma d\tau \quad (3)$$

In addition, there are in general equal-time contractions with factors $\delta(0)$ and $\theta(0)$. How should one evaluate such integrals? The function $\theta(\sigma - \tau) \theta(\tau - \sigma)$ is nonvanishing only at $\sigma = \tau$, a set of zero measure, so any smoothing of $\delta(\sigma - \tau)$ while keeping the product of the two θ 's intact leads directly to zero. A more natural way would seem to be to expand $\delta(\sigma - \tau)$ and $\theta(\sigma - \tau)$ into an infinite series, and to cut off these series at some large N ('mode regularization'). (The series expansion of $\theta(\sigma - \tau)$ could for example be defined so that $\partial_\sigma \theta(\sigma - \tau) = \delta(\sigma - \tau)$). Performing the whole calculation for finite N , one would expect to obtain the correct answer by taking the limit $N \rightarrow \infty$ at the end of the calculation. This is incorrect as we shall see but first we must answer a fundamental question: what does the expression 'the correct answer' mean?

Several authors have tried to give meaning to continuum path integrals in curved space, in particular configuration space integrals of the form $\int [dx^i] \exp(-\frac{1}{\hbar} S)$, by freely inventing definitions which maintain Einstein invariance at intermediate steps. The transition element $\langle z, \bar{\eta} | \exp(-\frac{\beta}{\hbar} \hat{H}) | y, \chi \rangle$ being in principle known from either heat kernel methods [7, 8] or direct operator approaches⁹, the loop calculation based on these path integrals should in the end reproduce the results for the transition element. Here problems arise: no prescription for L is known which keeps covariance at all stages and which gives the correct result at two- or higher loops.

Our point of view is the following: we define the continuum limit path integral as the limit of the discretized path integral obtained from (1). Hence, for us 'the correct answer' means: the answer which reproduces the results of the Hamiltonian approach. Integrals over products of discretized propagators, vertices and equal-time contractions are well defined and finite and by *taking the continuum limit, the correct Feynman rules automatically emerge*. The results are that $\delta(\sigma - \tau)$ should still be considered as a Kronecker delta function, even in the continuum case. So, for example, the correct value of I in (3) is 1/4. To bring out the surprising consequences of these new Feynman rules, consider another integral

$$I = \int_{-1}^0 \int_{-1}^0 \delta(\sigma - \tau) \theta(\sigma - \tau) \theta(\sigma - \tau) d\sigma d\tau \quad (4)$$

⁹Writing $\langle z | \exp(-\frac{\beta}{\hbar} \hat{H}) | y \rangle$ as $\int \langle z | \exp(-\frac{\beta}{\hbar} \hat{H} | p \rangle \langle p | y \rangle d^n p$, it is clear that by expanding the exponent and moving all \hat{p}_i to the right and all \hat{x}^i to the left, keeping track of commutators, coefficients of a given power of β are finite and unambiguous. Similarly when fermions are present.

If one were to require that $\delta(\sigma - \tau) = \partial_\sigma \theta(\sigma - \tau)$ even at the regularized level, one would find $1/3$ for this integral, whereas the correct answer (and the answer obtained from treating $\delta(\sigma - \tau)$ as a Kronecker delta) is $1/4$. This demonstrates that mode regularization is incorrect.

No expressions with higher powers of $\delta(\sigma - \tau)$ arise if one introduces in configuration space path integrals the extra ghosts of [9]. These ghosts are necessary to exponentiate the factors $g^{1/2}$ which arise if one integrates out the momenta. These factors $g^{1/2}$ were first found by Lee and Yang in a study of non-linear deformations of the harmonic oscillator [10] and were written by them as extra terms in the action of the form $g^{1/2}\delta(0)$. We find it much more convenient for higher loop calculations to replace them by local terms with ghosts, similarly to the familiar Faddeev-Popov ghosts in gauge theories. The phase space path integrals are always finite since the propagators do not have any $\delta(\sigma - \tau)$ singularities, and with these ghosts also the configuration space path integrals are finite¹⁰.

In the first part of this article we shall give a careful derivation of the new Feynman rules. We begin by *rewriting* the Hamiltonian \hat{H} in Weyl-ordered form $(H)_W$, and then use that matrix elements of $\exp(-\frac{\epsilon}{\hbar}(H)_W)$ with $\epsilon = \beta/N$ can be immediately evaluated to order ϵ by using the ‘midpoint rule’. (Weyl ordering is discussed in section 2.3.) Namely one may replace $\exp(-\frac{\epsilon}{\hbar}\hat{H})$ by $\exp(-\frac{\epsilon}{\hbar}(H)_W)$ in the kernels of the path integral

$$\int \langle x_k, \bar{\eta}_k | \exp(-\frac{\epsilon}{\hbar}\hat{H}) | p_k, \xi_{k-1} \rangle e^{-\bar{\xi}_{k-1}\xi_{k-1}} \langle \bar{\xi}_{k-1}, p_k | x_{k-1}, \eta_{k-1} \rangle dp_{k,i} d\bar{\xi}_{k-1,a} d\xi_{k-1}^a \quad (5)$$

and then one may replace in $(H)_W$ the operator \hat{p}_i by $(p_k)_i$, \hat{x}^i by $\frac{1}{2}(x_k^i + x_{k-1}^i)$, $\hat{\psi}_a^\dagger$ by $\bar{\xi}_{k-1,a}$, and $\hat{\psi}^a$ by $\frac{1}{2}(\xi_{k-1}^a + \eta_{k-1}^a)$. (Or $\hat{\psi}_a^\dagger$ by $\frac{1}{2}(\bar{\eta}_{k,a} + \bar{\xi}_{k-1,a})$ and $\hat{\psi}^a$ by ξ_{k-1}^a .) The net effect is that one can extract the function $H(p_k, \frac{1}{2}(x_k + x_{k-1}), \bar{\xi}_{k-1}, \frac{1}{2}(\xi_{k-1} + \eta_{k-1}))$. For linear sigma models (with $H = T + V$) rigorous proofs based on Banach spaces and the ‘Trotter formula’ [11] exist [4]. These do not apply to non-linear sigma models, but we have found a simple derivation of (5) which is precise enough for our taste. Note that no matter whether \hat{H} is gauge invariant or not, this midpoint rule holds; it is a purely algebraic result.

By using the background field formulation and coupling quantum deviations to external sources and decomposing the action S in a suitable free part $S^{(0)}$ and an interaction part, we find discretized propagators and vertices in closed form. The

¹⁰Power counting would seem to indicate that there are linear divergences due to the double derivative interactions. This would seem to contradict the theorem that quantum mechanics is finite. The ghosts save the theorem.

bosonic discretized propagators were already found in [1], while fermionic propagators for Dirac fermions were already found in [12].

We shall first consider Dirac fermions, but then an even number of Majorana (real) fermions $\psi^a(t)$ and operators $\hat{\psi}^a$ satisfying the Dirac brackets $\{\hat{\psi}^a, \hat{\psi}^b\} = \delta^{ab}$. To define coherent states we need creation and annihilation operators, and these we shall construct in two different ways: by doubling the number of fermions by adding a second set ψ_2^a of free fermions, or by ‘halving the number of fermions’ and constructing ψ^A and $\bar{\psi}_A$ from pairs ψ^{2a-1} and ψ^{2a} as $(\psi^{2a-1} \pm i\psi^{2a})/\sqrt{2}$. In either case we Weyl order, use the fermionic midpoint rule, and find propagators. The propagators $\langle \psi^a(\sigma)\psi^b(\tau) \rangle$ are different depending on whether one doubles or halves the Majorana fermions, and also the action in the path integral and the transition elements differ. In applications to anomalies, however, these differences disappear. In the conclusions we explain the reason for these results.

In all cases (bosonic systems, Dirac fermions, Majorana fermions either with doubling or halving), the propagators can be factored as follows

$$\langle z, \bar{\eta} | \exp \left(-\frac{\beta}{\hbar} \hat{H} \right) | y, \chi \rangle = e^{-\frac{1}{\hbar} S_{\text{class}}} \tilde{D}_S^{1/2} e^{\mathcal{A}_n} \quad (6)$$

where S_{class} is the classical action, $\tilde{D}_S^{1/2}$ contains the Van Vleck determinant and takes care of the one-loop contributions, while to order β \mathcal{A}_n contains the trace anomaly for $n = 2$ dimensions (it is proportional to the scalar curvature R). This structure of the propagator was proven for general bosonic systems in [7]. For fermionic systems \tilde{D}_S is actually the superdeterminant of $-\frac{\partial}{\partial \Phi^I} S_{\text{class}} \frac{\overleftarrow{\partial}}{\partial \Phi_J}$ where $\Phi^I = \{z^i, \bar{\eta}_a\}$ and $\Phi_J = \{y^i, \chi_a\}$. The fact that the one-loop contributions should be equal to the Van Vleck superdeterminant is a check on the correctness of our Feynman rules.

In general, the action in the path integral contains extra terms of order \hbar and \hbar^2 . These extra terms are due to rewriting the Hamiltonian in Weyl ordered form. In our case, we shall only encounter \hbar^2 terms. Hints that such terms might be necessary in the path integral were first found by DeWitt [7]. Schwinger, who studied the Poincaré operator algebra for Yang-Mills theory in the Coulomb gauge (which is a non-linear sigma model in 4 dimensions), found that one had to add extra non-naive terms of higher order in \hbar to the Hamiltonian in order that the algebra closes [13]. Subsequently many others have studied these extra terms [14, 15]. To check our new Feynman rules and also check that no further modifications at order \hbar^3 are present, we perform a 3-loop calculation in appendix A.1.

Our methods also apply to systems with more than two momenta (higher derivative theories). We consider in appendix A.2 such a system, and check that the results

of the phase space approach agree with those of the configuration space approach. This is Matthews' theorem [16]. It provides another test on the correctness of our results.

The second part of this article contains applications of these quantum mechanical path integrals to anomalies. Anomalies of an n -dimensional field theory can according to Fujikawa be written as [17]

$$\mathcal{A}_n = \text{Tr}(\hat{J}e^{-\beta\hat{R}}) \quad (7)$$

where \hat{J} is the Jacobian for a symmetry transformation of the path integral, and \hat{R} is a regulator. A general method to construct consistent regulators \hat{R} which maintain a given set of symmetries is given in [18]. It was first proposed by Alvarez-Gaumé and Witten [2] to consider a corresponding linear or non-linear sigma model in one dimension, for which \hat{R} becomes the Hamiltonian \hat{H} . The basic idea is that the Fujikawa trace can be viewed as the trace over the Hilbert space of a quantum mechanical system with a finite number of operators $x^\mu, \partial/\partial x^\mu$, Dirac matrices γ^a and, if present, internal symmetry generators T_a . After representing the Dirac matrices and internal symmetry generators (by means of Majorana fermions and an auxiliary ghost system) as operators in the same Hilbert space, the Fujikawa trace can be rewritten in terms of a suitable quantum mechanical path integral. The path integral can be used to compute matrix elements, and in terms of those the anomaly becomes

$$\mathcal{A}_n = \int dx^i \sqrt{g(x)} d\eta^a d\bar{\eta}_a e^{\bar{\eta}_a \eta^a} \langle x, \bar{\eta} | J e^{-\frac{\beta}{\hbar} H} | x, \eta \rangle \quad (8)$$

and by inserting a complete set of states between J and $\exp(-\beta H/\hbar)$ one obtains a product of the transition element and the matrix element of J . The anomaly is the β -independent term. Depending on the anomaly, i.e., depending on the matrix element of J , there are different factors of β in front of this expression, and since β counts the number of loops, different anomalies require a different number of world-line loops to be evaluated.

The simplest anomalies are the chiral anomalies. For these (8) is actually β independent, due to the topological nature of chiral anomalies, and, as we shall show, as a result one only needs to evaluate one-loop or tree graphs. In fact, Alvarez-Gaumé and Witten wrote the whole expression in (8) again as a path integral of the same kind as we consider for the transition element, but now with periodic boundary conditions both for the bosons and for the fermions. The one-loop contribution for such path integrals can then easily be written as the determinant of the kinetic operator of deviations about classical solutions. We shall obtain, of course, the same results but

will start from the transition element we obtained previously, and then simply do the rest of the integrals in (8). In our approach the ghosts for internal symmetries are part of the complete path integral, and are not treated by operator methods and by projecting on one-particle states as in [2]. The results of Alvarez-Gaumé and Witten were extended to trace anomalies in [9]. In this case the product of the Jacobian and the transition element could again be written as a path integral with now anti-periodic boundary conditions for the fermions, but since the background fermions are constant if they satisfy the equations of motion, the authors of [9] had problems in finding the correct boundary conditions at $t = -\beta$ and $t = 0$ for Majorana fermions. Rather, they used an operator formalism for the fermions, and operator-valued actions. We shall present a complete path integral formulation. Since the trace anomaly receives contributions from higher loop graphs, the details of the path integral do matter very much. For example, forgetting the extra \hbar^2 terms in the action or the measure factor, one obtains incorrect results. We shall also give a derivation of trace anomalies. Although we will not do so here, one can use the same framework to derive the gravitational anomalies due to spin-1/2 and spin-3/2 fields. The only subtlety is the question precisely which Jacobian one should use in (7) for spin-3/2, and this is discussed in detail in appendix A.3.

We conclude this introduction by stating which of our results are new. The Feynman rules for products of distributions¹¹ are new; for the bosonic case they were obtained in [1] and for the fermionic case here. Our treatment of coherent states for fermions (first considered in [20]) follows [21] and is somewhat simpler (we believe) than those treatments in the literature which use four kinds of coherent states (namely bras and kets which are eigenstates of $\hat{\psi}$ or $\hat{\psi}^\dagger$). For a good discussion of the latter see [22]. The careful treatment of Majorana fermions (doubling and halving), in particular the fact that the transition elements are different, is new, as is the proof that in all cases the one-loop contributions sum up to a superdeterminant. New in section three is the complete path integral treatment of chiral and trace anomalies, with no need to introduce matrix-valued Hamiltonians or to perform certain projections on the ghost states by hand. Also new is the complete diagrammatic evaluation of these anomalies in section 3 and appendix A.4. The final new result is the correct incorporation of ‘Lee-Yang’ ghosts for higher derivative theories in appendix A.2.

Besides all these new results, we have spent a great deal of time to make the whole

¹¹Mathematically, it is possible to multiply objects called generalized functions that contain the set of distributions. However, to determine which generalized function corresponds to a given distribution still requires an ‘underlying physical principle’ (see e.g. [19] and references therein), and our rules can be seen as an example of such a principle.

subject of quantum mechanical path integrals for non-linear sigma models accessible to a large audience, and we hope that this article will also be a useful review of known results.

2 Path integrals for finite time

In this section we establish the framework we need in order to perform calculations using one-dimensional path integrals for finite time. Given a quantum mechanical system defined by a Hamiltonian with a certain ordering prescription, we derive the corresponding path integral formulation. This includes both the action to be used in the path integral, as well as the Feynman rules, the latter being not only a set of expressions for the propagators and vertices, but also the correct prescription how to evaluate integrals over products of these, as they occur in actual loop calculations. We will first discuss the bosonic non-linear sigma model, and derive the correct rules for configuration space path integrals. For phase space path integrals for bosonic systems, see [1]. Then we will derive similar results for the extension to complex (Dirac) fermions, for which we will need to introduce coherent states. Finally we shall discuss the modifications which one must make for Majorana fermions. Along the way, we will present a variety of checks that our Feynman rules are the correct ones, by evaluating various transition elements and comparing these with the results obtained from operator methods. Further convincing evidence is provided in appendix A.1 and A.2.

2.1 Bosonic non-linear sigma model

We will start by computing the transition element for the following quantum mechanical Hamiltonian

$$\hat{H} = \frac{1}{2} g^{-1/4} p_i \sqrt{g} g^{ij} p_j g^{-1/4} \quad (9)$$

This operator is Einstein invariant if p_i is hermitian. (The p_i transform under Einstein (=general co-ordinate) transformations as $p'_i = \frac{1}{2} \left\{ \frac{\partial x^j}{\partial x'^i}, p_j \right\}$, if the inner product is defined by (1), from which the Einstein invariance of \hat{H} follows [3]). The more general case, where also a scalar and vector potential are present, can easily be found at each stage in the computation by covariantizing the expressions, and including the scalar potential in the interactions. The cases of other Hamiltonians, for example Hamiltonians whose operator ordering is different from (9), will also be clear.

One can evaluate the corresponding transition element,

$$T(z, y; \beta) \equiv \langle z | \exp \left(-\frac{\beta}{\hbar} \hat{H} \right) | y \rangle, \quad (10)$$

for finite (Euclidean) time β through a direct, but rather tedious, computation in the operator formalism [23, 1, 24]. Namely, by writing T as $\int d^n p \langle z | \exp(-\frac{\beta}{\hbar} \hat{H}) | p \rangle \langle p | y \rangle$ and expanding the exponent, moving all \hat{p}_i to the right and all \hat{x}^i to the left, keeping track of all terms with up to two commutators, we find the following result (correct through order β , counting $z - y$ as being of order $\beta^{1/2}$)

$$T = (2\pi\hbar\beta)^{-n/2} \exp \left(-\frac{1}{\hbar} S_{cl}^B[z, y; \beta] \right) \tilde{D}^{1/2} \exp \left(-\frac{\beta\hbar}{12} R \right) \quad (11)$$

where $(2\pi\hbar\beta)^{-n/2}$ is the usual Feynman factor, $S_{cl}^B[z, y; \beta]$ is the classical action for a geodesic with $x(-\beta) = y$, $x(0) = z$, $-\frac{1}{12}R$ yields the trace anomaly in $n = 2$ dimensions¹², and $\tilde{D}^{1/2}$ is proportional to the square root of the Van Vleck determinant [25] which gives the one-loop corrections to the transition element

$$\begin{aligned} \tilde{D}^{1/2} &= \beta^{n/2} g^{-1/4}(z) \det \left(-\frac{\partial}{\partial z^i} \frac{\partial}{\partial y^j} S_{cl}[z, y; \beta] \right)^{1/2} g^{-1/4}(y) \\ &= 1 - \frac{1}{12} R_{ij}(z) (z - y)^i (z - y)^j + \mathcal{O}(\beta^{3/2}) \end{aligned} \quad (12)$$

The results in (11) and (12) agree with DeWitt's classic paper [7] in which he uses heat kernel methods. Note that if one views T for $\beta \rightarrow 0$ as the kernel of a continuum path integral action, then $\tilde{D}^{1/2}$ corresponds to a non-local term. Each of the factors in (11) is a general co-ordinate bi-scalar (a scalar both in y and in z). We stress that the answer for T in (11) is finite and unambiguous.

We will now construct the path integral whose loop expansion reproduces (11). First we rewrite the Hamiltonian in Weyl-ordered form [26], which for any monomial in p and x is defined by $(n + m)!(p^n x^m)_W = \partial_a^n \partial_b^m (a\hat{p} + b\hat{x})^{m+n}$ [27, 28, 29]. For the Hamiltonian in (9) we find the well known result [27, 30]

$$\hat{H} = \left(\frac{1}{2} g^{ij} p_i p_j \right)_W + \frac{\hbar^2}{8} \left(g^{ij} \Gamma_{il}^k \Gamma_{jk}^l + R \right) \quad (13)$$

Then we use the correspondence between Weyl-ordering and the midpoint rule [31, 27]

$$\int d^n p \langle x_{k+1} | G_W | p \rangle \langle p | x_k \rangle = \int d^n p G(p, x_{k+1/2}) \langle x_{k+1} | p \rangle \langle p | x_k \rangle \quad (14)$$

¹² Our convention for the curvatures are $R(\Gamma)_{\rho\sigma\mu}{}^\nu = \partial_\rho \Gamma_{\sigma\mu}{}^\nu + \Gamma_{\rho\tau}{}^\nu \Gamma_{\sigma\mu}{}^\tau - (\rho \leftrightarrow \sigma) = R(\omega)_{\rho\sigma a}{}^b e_\mu^b e_a^\nu$ with $R_{\rho\sigma ab} = \partial_\rho \omega_{\sigma ab} + \omega_{\rho a}{}^c \omega_{\sigma cb} - (\rho \leftrightarrow \sigma)$ and $R_{\mu\nu} = R(\Gamma)_{\mu\sigma\nu}{}^\sigma$, so $R_{\mu b} = R(\omega)_{\mu\rho ab} e^{a\rho}$, while $R = g^{\mu\nu} R_{\mu\nu} = (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \partial_\mu \partial_\nu g_{\rho\sigma} + \dots$

where we defined $x_{k+1/2} = (x_k + x_{k+1})/2$, and G is any function of p, x . Clearly, $\exp\left(-\frac{\epsilon}{\hbar}H_W\right) = (\exp\left(-\frac{\epsilon}{\hbar}H\right))_W + \mathcal{O}(\epsilon^2)$, so using (14) we find

$$\begin{aligned} \int d^n p \langle x_{k+1} | \exp\left(-\frac{\epsilon}{\hbar}H_W\right) | p \rangle \langle p | x_k \rangle \\ = \int d^n p \exp\left(-\frac{\epsilon}{\hbar}H(p, x_{k+1/2})\right) \langle x_{k+1} | p \rangle \langle p | x_k \rangle + \mathcal{O}(\epsilon^2) \end{aligned} \quad (15)$$

Note that $H(p, x_{k+1/2})$ in (15) contains the terms in (13) of order \hbar^2 . We argued in [1] that the terms denoted by $\mathcal{O}(\epsilon^2)$ will not contribute to the path integral, and can therefore be neglected. This is the only point in our derivation of the path integral that is not mathematically completely rigorous. What is subtle is the meaning of $\mathcal{O}(\epsilon^2)$. For example, one can view p either as being of order 1 or of order $\epsilon^{-1/2}$ (since there are Gaussian integrals with $\exp(-p^2/2\epsilon)$). In the latter case the terms denoted by $\mathcal{O}(\epsilon^2)$ are actually of order ϵ . Each case leads to a different kernel. We argued in [1], using the effective potential trick of [14] and [32], that both kernels are equivalent under the path integral. However, this is not a rigorous argument. Ideally one should keep all terms on the right hand side of (15) that might contribute in the limit $N \rightarrow \infty$, evaluate the path integral at the discretized level and then prove that in the limit $N \rightarrow \infty$ all extra terms do indeed drop out.

We now insert $N - 1$ sets of x -eigenstates and N sets of p -eigenstates into $\langle z | \exp -\frac{\beta}{\hbar} \hat{H} | y \rangle$, and we arrive at the discretized phase space path integral using $\int d^n x \sqrt{g(x)} |x\rangle \langle x| = 1 = \int d^n p |p\rangle \langle p|$ and $\langle z | p \rangle = (2\pi\hbar)^{-n/2} (\exp \frac{i}{\hbar} p \cdot z) g^{-1/4}(z)$. Integrating out the N momenta we find the discretized configuration space path integral, with N factors $g^{1/2}(x_{k+1/2})$ in the measure from the p integrals, N products $g^{-1/4}(x_{k+1})g^{-1/4}(x_k)$ from the inner products $\langle x | p \rangle$ and $N - 1$ factors $g^{1/2}(x_k)$ from the completeness relation in x -space. The action is given by

$$S = \sum_{k=0}^{N-1} \left[\frac{1}{2\epsilon} g_{ij}(x_{k+1/2}) (x_{k+1} - x_k)^i (x_{k+1} - x_k)^j + \frac{\hbar^2 \epsilon}{8} (\Gamma\Gamma + R)(x_{k+1/2}) \right] \quad (16)$$

where we define $x_N = z$ and $x_0 = y$, and $\epsilon = \beta/N$. We decompose x_k^j into a background and a quantum part, and S into a free and interacting part

$$x_k^j = x_{\text{bg},k}^j + q_k^j, \quad S = S^{(0)} + S^{(\text{int})}; \quad k = 1, \dots, N-1 \quad (17)$$

where $S^{(0)} = \sum_{k=0}^{N-1} \frac{1}{2\epsilon} g_{ij}(z) (q_{k+1} - q_k)^i (q_{k+1} - q_k)^j$. We take the metric in $S^{(0)}$ at z in order to facilitate comparison with (11), although any other choice should give the same result. (Of course, propagators and vertices will be different if we make a different decomposition into a kinetic and interaction part, and also the measure factor (see (22)) will be different, but the final results should not change). Since

we take x_{bg} to be a solution of the classical equations of motion of $S^{(0)}$, in general $S^{(\text{int})}$ contains in addition to the true interactions also a pure background piece and terms linear in q_k^j . However, counting $z - y$ as being of order $\beta^{1/2}$, we need only a finite number of tree graphs and tadpoles at a given order in β . The N factors $g^{1/2}(x_{k+1/2})$ are exponentiated following [9] (for an alternative approach see [33]) by using anti-commuting ghosts b and c and a commuting ghost a

$$\sqrt{\det g_{ij}(x_{k+1/2})} = K \int db_{k+1/2}^j dc_{k+1/2}^j da_{k+1/2}^j \exp \left(-\frac{\epsilon}{2\beta^2 \hbar} g_{ij}(x_{k+1/2}) (b_{k+1/2}^i c_{k+1/2}^j + a_{k+1/2}^i a_{k+1/2}^j) \right) \quad (18)$$

Since the constant K will cancel, we do not determine it, and the reason for the particular normalization of the ghost action will become clear later. Introducing modes for the quantum fluctuations q by the orthonormal transformation

$$q_k^j = \sum d_m^j \sqrt{\frac{2}{N}} \sin \left(\frac{km\pi}{N} \right); \quad k, m = 1, \dots, N-1 \quad (19)$$

we may change $dx_k^j \rightarrow dq_k^j \rightarrow dd_m^j$. Obviously, the Jacobian for this transformation is 1. The quantum part of the action $S^{(0)}$ becomes equal to

$$-\frac{1}{\hbar} S^{(0)}(q) = -\sum_{m=1}^{N-1} \frac{1}{\epsilon \hbar} g_{ij}(z) d_m^i d_m^j (1 - \cos \frac{m\pi}{N}). \quad (20)$$

Next, we couple to external sources

$$S^{(\text{source})} = -\epsilon \sum_{k=0}^{N-1} \left(F_{k+1/2} \frac{q_{k+1} - q_k}{\epsilon} + G_{k+1/2} q_{k+1/2} + \text{sources for ghosts} \right) \quad (21)$$

so that we can extract the exact discretized propagators in the usual way. Completing squares and performing the final integration over d_m^j , $b_{k+1/2}^j$, $c_{k+1/2}^j$, $a_{k+1/2}^j$ leads to $N-1$ factors $g^{-1/2}(z)$ and N factors $g^{1/2}(z)$ as well as an overall factor $(2\pi\hbar\beta)^{-n/2}$. (The factor $(2\pi\hbar\epsilon)^{-Nn/2}$ which comes from the p integrations and the normalization of the plane waves combines with the factor $\prod_{m=1}^{N-1} (\pi\epsilon\hbar)^{n/2} (1 - \cos \frac{m\pi}{N})^{-n/2}$ from the Gaussian integrations over d_m^j to yield $(2\pi\hbar\beta)^{-n/2}$ since $\prod_{m=1}^{N-1} 2(1 - \cos \frac{m\pi}{N}) = N$). Hence

$$T = \left(\frac{g(z)}{g(y)} \right)^{1/4} (2\pi\hbar\beta)^{-n/2} \exp(-\frac{1}{\hbar} S^{(\text{int})}) \exp(-\frac{1}{\hbar} S^{(\text{source})}). \quad (22)$$

The measure factor $(g(z)/g(y))^{1/4}$ is due to the split $S = S^{(0)} + S^{(\text{int})}$ where in $S^{(0)}$ the metric is taken at z , $g_{ij}(z)$ (cf. question (iii) in the introduction). Different splits clearly lead to different measures, but we continue with (22). Such factors are often ignored but are crucial to obtain the correct transition element. Equation

(22) is to be read as usual in path integral formulations, namely $S^{(\text{int})}$ contains only derivatives with respect to the sources and background fields, while $S^{(\text{source})}$ is the function bilinear in sources that appears after doing all the integrals, and in the final result we are supposed to put all the sources equal to zero. Thus, $S^{(\text{int})}$ contains the discretized vertices while $S^{(\text{source})}$ yields the discretized propagators. Defining $\dot{x}_{k+1/2} = (x_{k+1} - x_k)/\epsilon$ and omitting superscripts and a factor of $\hbar g^{ij}(z)$ for the time being, the propagators come out as follows

$$\begin{aligned}
\langle q_{k+1/2} q_{l+1/2} \rangle &= -\frac{\epsilon}{4N} (2k+1)(2l+1) + \frac{\epsilon}{2} (2 \min(k, l) + 1 - \frac{1}{2} \delta_{k,l}) \\
\langle q_{k+1/2} \dot{q}_{l+1/2} \rangle &= -\frac{k+1/2}{N} + \theta_{k,l} \\
\langle \dot{q}_{k+1/2} \dot{q}_{l+1/2} \rangle &= -\frac{1}{N\epsilon} + \frac{1}{\epsilon} \delta_{k,l} \\
\langle b_{k+1/2} c_{l+1/2} \rangle &= -\frac{2}{\epsilon} \delta_{k,l} \\
\langle a_{k+1/2} a_{l+1/2} \rangle &= \frac{1}{\epsilon} \delta_{k,l},
\end{aligned} \tag{23}$$

where $\theta_{k,l}$ is a discretization of the θ function: $\theta_{k,l} = 0$ if $k < l$, $\theta_{k,l} = 1/2$ if $k = l$ and $\theta_{k,l} = 1$ if $k > l$.

As an example, let us give a more detailed derivation of the $\langle q_{k+1/2} q_{l+1/2} \rangle$ propagator. The partition function (we set the sources F equal to zero, and suppress internal indices) reads

$$Z[G] = \int \prod_{i=1}^{N-1} dq_i \exp -\frac{\epsilon}{\hbar} \left[\sum_{k=0}^{N-1} \frac{1}{2} \left(\frac{q_{k+1} - q_k}{\epsilon} \right)^2 + G_{k+1/2} \left(\frac{q_{k+1} + q_k}{2} \right) \right] \tag{24}$$

Now make a change of variables as defined in (19) and (20), complete the squares and do the Gaussian integrals over the modes d_m . Up to an overall numerical factor which was already taken care of in (22), the result equals

$$\begin{aligned}
Z[G] \sim \exp & \left[\frac{\epsilon^3}{2N\hbar} \sum_{j=1}^{N-1} \frac{1}{1 - \cos(j\pi/N)} \right. \\
& \left. \left(\sum_{k=0}^{N-1} G_{k+1/2} \sin((k+1/2)j\pi/N) \cos(j\pi/(2N)) \right)^2 \right]
\end{aligned} \tag{25}$$

Differentiating with respect to the sources G gives rise to the following expression for the propagator

$$\begin{aligned}
\langle q_{k+1/2} q_{l+1/2} \rangle &= \frac{\epsilon\hbar}{2N} \sum_{j=1}^{N-1} \left[\cos^2(j\pi/(2N)) \frac{\sin((k+1/2)j\pi/N)}{\sin(j\pi/(2N))} \right. \\
& \quad \left. \frac{\sin((l+1/2)j\pi/N)}{\sin(j\pi/(2N))} \right]
\end{aligned} \tag{26}$$

Each term can be written as a sum of powers of $e^{i\pi/(2N)}$, and performing the sum over j yields the propagator as given in (23). The remaining discretized propagators can be found in a similar fashion. We require that $x_{\text{bg},k}^j$ satisfies the boundary conditions and the equation of motion of $S^{(0)}$. In the continuum limit this becomes $x_{\text{bg}}^j(t) = z^j + (z - y)^j t / \beta$, while $q^j(t)$ vanishes at the endpoints. In this limit the two-point functions become (reinstating the superscripts, factors of $g^{ij}(z)$, and defining $t = \beta\tau$)

$$\begin{aligned}
\langle q^i(\sigma) q^j(\tau) \rangle &= -\beta \hbar g^{ij}(z) \Delta(\sigma, \tau) \\
\langle b^i(\sigma) c^j(\tau) \rangle &= -2\beta \hbar g^{ij}(z) \partial_\sigma^2 \Delta(\sigma, \tau) \\
\langle a^i(\sigma) a^j(\tau) \rangle &= \beta \hbar g^{ij}(z) \partial_\sigma^2 \Delta(\sigma, \tau) \\
\Delta(\sigma, \tau) &= \sigma(\tau + 1)\theta(\sigma - \tau) + \tau(\sigma + 1)\theta(\tau - \sigma).
\end{aligned} \tag{27}$$

Note that $\Delta(\sigma, \tau) = \Delta_F(\sigma - \tau) + \sigma\tau + \frac{1}{2}(\sigma + \tau)$, where $\Delta_F(\sigma - \tau) = \frac{1}{2}(\sigma - \tau)\theta(\sigma - \tau) + \frac{1}{2}(\tau - \sigma)\theta(\tau - \sigma)$ is the Feynman propagator, and formally $\partial_\sigma^2 \Delta(\sigma, \tau) = \delta(\sigma - \tau)$ while $\Delta(\sigma, \tau) = 0$ at the boundaries. However, *the $\delta(\sigma - \tau)$ is a Kronecker delta and moreover the equal-time contractions¹³ are unambiguously defined.* Kronecker delta here means that $\int dx \delta(x) f(x) = f(0)$, even when f contains a product of θ functions. From (23) we further find in the continuum limit

$$\begin{aligned}
\langle q^i(\sigma) \dot{q}^j(\tau) \rangle &= -\beta \hbar g^{ij}(z) (\sigma + \theta(\tau - \sigma)) \\
\langle \dot{q}^i(\sigma) \dot{q}^j(\tau) \rangle &= -\beta \hbar g^{ij}(z) (1 - \delta(\sigma - \tau)).
\end{aligned} \tag{28}$$

All propagators are now proportional to $\beta \hbar$ (this motivated the normalization of the ghost action in (18)), and the interactions are given by

$$\begin{aligned}
\frac{1}{\hbar} S^{(\text{int})} &= \frac{1}{\beta \hbar} \int_{-1}^0 \left[\frac{1}{2} g_{ij}(x_{\text{bg}} + q) \left\{ (\dot{x}_{\text{bg}} + \dot{q})^i (\dot{x}_{\text{bg}} + \dot{q})^j + b^i c^j + a^i a^j \right\} \right] d\tau \\
&\quad + \beta \hbar \int_{-1}^0 \frac{1}{8} (\Gamma \Gamma + R) d\tau - \frac{1}{\hbar} S^{(0)}, \\
\frac{1}{\hbar} S^{(0)} &= \frac{1}{\beta \hbar} \int_{-1}^0 \left[\frac{1}{2} g_{ij}(z) \left\{ \dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \right\} \right] d\tau
\end{aligned} \tag{29}$$

Clearly, the interactions only depend on the combination $\beta \hbar$.

To compute the configuration space path integral, we note that we must expand the measure factor $g^{1/4}(z)/g^{1/4}(y)$ in (22) and evaluate all vacuum graphs with external x_{bg} , using the propagators in (27), (28) and the vertices in (29). The q -independent part of $S^{(\text{int})}$ does not yield the full S_{cl}^B of (11) since x_{bg} is only a solution of the $S^{(0)}$ equation of motion; rather, tree graphs with two vertices from

¹³Equal-time contractions in quantum field theory can in general only be fixed by imposing a symmetry principle [34]. In our case they are fixed by our requirement that the path integral reproduces the Hamiltonian results.

$S^{(\text{int})}$ contribute to order β terms of the form $\frac{1}{\beta}(\partial g)^2(z-y)^4$, see (66). In the one-loop graphs with one vertex $S^{(\text{int})}$ one finds equal-time contractions proportional to $(z-y)^k \partial_k g_{ij}$ times $\langle \dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \rangle$ in which the $\delta(0)$ cancel, yielding $\int_{-1}^0 \int_{-1}^0 \sigma(\partial_\sigma \partial_\tau \Delta + \partial_\sigma^2 \Delta) d\sigma d\tau = -\frac{1}{2}$, which cancels a similar contribution from the non-trivial measure factor. There are many other one-loop and two-loop graphs, and the contribution of each corresponds to a particular term in (11). In particular, the two-loop graph with one $\dot{q}\dot{q}$, one $\dot{q}q$ and one $q\dot{q}$ propagator agrees with (11) only if one uses $\int_{-1}^0 \int_{-1}^0 \delta(\sigma-\tau) \theta(\sigma-\tau) \theta(\tau-\sigma) = \frac{1}{4}$, in agreement with the discretized expressions for the propagators in (23). Adding all contributions we have found complete agreement. The non-covariant vertices $\frac{\beta\hbar}{8}(\Gamma\Gamma + R)$ conspire with the non-covariant vertices found by expanding $g_{ij}(x)$ and yield the Einstein invariant expression (11). The Feynman rules one has to use in this calculation follow from (23), and they amount to the following. First, one writes down expressions for all Feynman diagrams using the propagators given by (27) and (28). Adding everything, all divergences coming from products of delta functions will cancel (the ghosts of [9] are crucial for this). The resulting integrals should be worked out using the rules that delta functions should really be seen as Kronecker deltas and that $\theta(0) = 1/2$. If there are explicit delta functions in the integrals, one should be careful with partial integrations and identities like $\int_a^b f' = f(b) - f(a)$, since these are not always compatible with our Kronecker delta prescription [1]. Luckily, in practice we never need to partially integrate.

2.2 The fermionic case

We now repeat the analysis of the last section for the fermionic case. We will work in a basis of coherent states, and the derivation of the path integral is analogous to the one in phase space for the case of the bosonic non-linear sigma model. We consider operators $\hat{\psi}^a, \hat{\psi}_a^\dagger$, $a = 1 \dots n$, satisfying the anticommutation relations $\{\hat{\psi}^a, \hat{\psi}_b^\dagger\} = \delta_b^a$. These operators $\hat{\psi}$ are obtained from the canonical variables by rescaling with a factor of $\hbar^{-1/2}$. As a consequence, terms of the form $\hbar^2 R \psi^4$ or $\hbar \omega \psi^2$ are terms of the classical action, not higher-loop terms. For cases such as the $N=1$ supersymmetric non-linear sigma model, where only Majorana fermions are present, we will need to replace these Majorana fermions by Dirac fermions. This will be discussed later, here we will derive the general expression for the path integral with Dirac fermions.

Coherent states are defined by

$$|\eta\rangle = e^{\hat{\psi}^\dagger \eta} |0\rangle \quad ; \quad \langle \bar{\eta} | = \langle 0 | e^{\bar{\eta} \hat{\psi}} \quad (30)$$

satisfying $\hat{\psi}|\eta\rangle = \eta|\eta\rangle$ and $\langle \bar{\eta} | \hat{\psi}^\dagger = \langle \bar{\eta} |$. We could in addition also define coherent

states build around a Dirac vacuum (completely-filled Fermi sea) $\langle \eta | = (-1)^n \langle 0 | \hat{\psi}^n \dots \hat{\psi}^1 (e^{\eta \hat{\psi}^\dagger})$, satisfying $\langle \eta | \hat{\psi}^a = \langle \eta | \eta^a$, and similarly $|\bar{\eta}\rangle = (e^{\bar{\psi}^k \bar{\eta}_k}) \hat{\psi}_1^\dagger \dots \hat{\psi}_n^\dagger |0\rangle (-)^n$. These states are often used in the literature for the construction of fermionic path integrals in a way which closely mimicks the $x \cdot p$ approach, see e.g. [22]. In contrast, in our approach we only use the coherent states (30); although both approaches are completely equivalent, we believe ours is more economical. The inner product and decomposition of unity read formally the same as for bosonic coherent states [21]

$$\langle \bar{\eta} | \xi \rangle = e^{\bar{\eta} \xi} \quad ; \quad 1 = \int d\bar{\eta} d\xi |\xi\rangle e^{-\bar{\eta} \xi} \langle \bar{\eta} | \quad (31)$$

but note the ordering of the anticommuting variables. Our convention is that $d\bar{\eta} = d\bar{\eta}_n \dots d\bar{\eta}_1$, while $d\xi = d\xi_1 \dots d\xi_n$, or equivalently, $d\bar{\eta} d\xi = \prod_{k=1}^n (d\bar{\eta}_k d\xi_k)$. Hence $\int d\xi \xi_n \dots \xi_1 = 1$, and $\int d\xi \prod_{k=1}^n \xi_k = (-1)^{n/2}$ for even n . With these conventions, the trace of an operator over the fermionic Fock space is given by

$$\text{trace}(A) = \int d\xi d\bar{\eta} e^{\bar{\eta} \xi} \langle \bar{\eta} | A | \xi \rangle \quad (32)$$

Again, we define Weyl-ordering by $(n+m)!(\psi^n \psi^{\dagger m})_W = \partial_{\bar{\eta}}^n \partial_{\eta}^m (\bar{\eta} \hat{\psi} + \eta \hat{\psi}^\dagger)^{m+n}$, with the fermionic derivatives acting from the left. For an arbitrary Weyl-ordered operator \hat{G} we can now derive the midpoint identity for coherent states

$$\begin{aligned} \langle \bar{\eta} | \hat{G} | \eta \rangle &= \int d\bar{\chi} d\chi G(\bar{\chi}, \frac{\chi + \eta}{2}) e^{-\bar{\chi} \chi} \langle \bar{\eta} | \chi \rangle \langle \bar{\chi} | \eta \rangle \\ &= \int d\bar{\chi} d\chi G(\frac{\bar{\chi} + \bar{\eta}}{2}, \chi) e^{-\bar{\chi} \chi} \langle \bar{\eta} | \chi \rangle \langle \bar{\chi} | \eta \rangle \\ &= \int d\bar{\chi} d\chi G(\lambda_1 \bar{\chi} + (1 - \lambda_1) \bar{\eta}, \lambda_2 \chi + (1 - \lambda_2) \eta) e^{-\bar{\chi} \chi} \langle \bar{\eta} | \chi \rangle \langle \bar{\chi} | \eta \rangle, \end{aligned} \quad (33)$$

with $\lambda_1 \lambda_2 = 1/2$. This formula can be proven in the following way (cf. [12]). It is easy to check that it is valid for an operator \hat{G} of the type $(\hat{\psi}^\dagger)^k$, which is automatically Weyl-ordered. Then one uses that, for a Weyl-ordered operator \hat{A} , the operator $\hat{B} = (\hat{\psi} \hat{A} \pm \hat{A} \hat{\psi})/2$ (the sign depending on whether \hat{A} is bosonic or fermionic) is also Weyl-ordered, and hence that any Weyl-ordered operator can be obtained by repeatedly applying this identity to an operator of the type $(\hat{\psi}^\dagger)^k$. We can now proceed by inserting unity in (33) for $\hat{G} = \hat{B}$, using its validity for \hat{A} as induction hypothesis. We find

$$\begin{aligned} \langle \bar{\eta} | \hat{B} | \eta \rangle &= \int d\bar{\chi} d\chi e^{-\bar{\chi} \chi} \frac{\chi + \eta}{2} \langle \bar{\eta} | \chi \rangle \langle \bar{\chi} | \hat{A} | \eta \rangle \\ &= \int d\bar{\chi} d\chi d\bar{\xi} d\xi e^{-\bar{\chi} \chi} \frac{\chi + \eta}{2} \langle \bar{\eta} | \chi \rangle A(\bar{\xi}, \frac{\eta + \xi}{2}) e^{-\bar{\xi} \xi} \langle \bar{\chi} | \xi \rangle \langle \bar{\xi} | \eta \rangle \\ &= \int d\bar{\xi} d\xi \frac{\xi + \eta}{2} A(\bar{\xi}, \frac{\xi + \eta}{2}) e^{-\bar{\xi} \xi} \langle \bar{\eta} | \xi \rangle \langle \bar{\xi} | \eta \rangle \\ &= \int d\bar{\xi} d\xi B(\bar{\xi}, \frac{\xi + \eta}{2}) e^{-\bar{\xi} \xi} \langle \bar{\eta} | \xi \rangle \langle \bar{\xi} | \eta \rangle \end{aligned} \quad (34)$$

where we have used that $\int d\bar{\chi} d\chi e^{-\bar{\chi}\chi} f(\chi) = f(0)$. This completes the proof of the first two lines of (33). The third line of (33) can be demonstrated in a similar fashion. We can now apply this identity to $\hat{G} = \exp\left(-\frac{\epsilon}{\hbar}\hat{H}_W\right)$, and use that \hat{G} is Weyl-ordered to order $\mathcal{O}(\epsilon^2)$. We find, neglecting these higher order terms, after inserting unity $N - 1$ times, with $\beta = N\epsilon$,

$$\begin{aligned} \langle \bar{\eta} | \exp\left(-\frac{\beta}{\hbar}\hat{H}_W\right) | \chi \rangle &= \\ \int \prod_{k=1}^{N-1} \left[d\bar{\xi}_k d\xi_k e^{-\bar{\xi}_k \xi_k} \langle \bar{\xi}_{k+1} | \exp\left(-\frac{\epsilon}{\hbar}\hat{H}_W\right) | \xi_k \rangle \right] \langle \bar{\xi}_1 | \exp\left(-\frac{\epsilon}{\hbar}\hat{H}_W\right) | \chi \rangle \end{aligned} \quad (35)$$

where we defined $\bar{\xi}_N = \bar{\eta}$. Now use the midpoint rule for each matrix element

$$\begin{aligned} \langle \bar{\xi}_{k+1} | \exp\left(-\frac{\epsilon}{\hbar}\hat{H}_W\right) | \xi_k \rangle &= \\ \int d\bar{\psi}_k d\psi_k e^{-\bar{\psi}_k \psi_k + \bar{\xi}_{k+1} \psi_k + \bar{\psi}_k \xi_k} \exp\left(-\frac{\epsilon}{\hbar}H\left(\bar{\psi}_k, \frac{\psi_k + \xi_k}{2}\right)\right) \end{aligned} \quad (36)$$

We can integrate out the $\bar{\xi}, \xi$ in (35) (first the $\bar{\xi}$, then the ξ) to obtain

$$\begin{aligned} \langle \bar{\eta} | \exp\left(-\frac{\beta}{\hbar}\hat{H}_W\right) | \chi \rangle &= \int \prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k \exp\left[\bar{\eta}\psi_{N-1} \right. \\ &\quad \left. - \epsilon \sum_{k=0}^{N-1} \left(\bar{\psi}_k \frac{\psi_k - \psi_{k-1}}{\epsilon} + \frac{1}{\hbar} H\left(\bar{\psi}_k, \frac{\psi_k + \psi_{k-1}}{2}\right) \right) \right], \end{aligned} \quad (37)$$

where we defined $\psi_{-1} = \chi$. The first term in the integrand is the usual boundary term one obtains in path integrals for coherent states (cf. [21, 35]). From this result we conclude that the action in the continuum path integral is $\int_{-\beta}^0 (\hbar \bar{\psi} \dot{\psi} + H) dt - \hbar \bar{\psi}(0) \psi(0)$ with the boundary conditions $\bar{\psi}(0) = \bar{\eta}$ and $\psi(-\beta) = \chi$. Notice that the boundary term is essential to produce the correct equations of motion $\dot{\bar{\psi}} = \dot{\psi} = 0$. After decomposing ψ_k and $\bar{\psi}_k$ into a background piece and a quantum piece, we couple the latter to external sources. Putting all of H into H^{int} , we obtain the discretized $\bar{\psi}\psi$ propagator by inverting the kinetic term matrix $A_{j,k} = \delta_{j,k} - \delta_{j,k+1}$. The result reads

$$\langle \bar{\psi}_k \psi_l \rangle = \begin{cases} -1 & k \leq l \\ 0 & k > l \end{cases} = -\theta_{l,k} - \frac{1}{2}\delta_{k,l}. \quad (38)$$

If we now define $\dot{\psi}_{k-1/2} = (\psi_k - \psi_{k-1})/\epsilon$ and $\psi_{k-1/2} = (\psi_k + \psi_{k-1})/2$, we obtain

$$\begin{aligned} \langle \bar{\psi}_k \psi_{l-1/2} \rangle &= -\theta_{l,k} \\ \langle \bar{\psi}_k \dot{\psi}_{l-1/2} \rangle &= -\frac{1}{\epsilon} \delta_{k,l} \end{aligned} \quad (39)$$

where we recall that $\theta_{k,l} = 0$ if $k < l$, $\theta_{k,l} = 1/2$ if $k = l$ and $\theta_{k,l} = 1$ if $k > l$. We can now, as in the bosonic case, write down the corresponding continuum expressions

which we will use in actual computations. However, it is important to realize that in diagrams in which products of δ and θ functions arise, we are now able to resolve the resulting ambiguities by returning to the discretized expressions (39). In particular, we find again the ‘rules’ that $\theta(\sigma, \sigma) = \frac{1}{2}$, and $\int d\sigma d\tau \theta(\sigma - \tau) \theta(\tau - \sigma) \delta(\sigma - \tau) = \frac{1}{4}$. The continuum propagators read

$$\begin{aligned} \langle \bar{\psi}^a(\sigma) \psi^b(\tau) \rangle &= -\delta^{ab} \theta(\tau - \sigma) \\ \langle \bar{\psi}^a(\sigma) \dot{\psi}^b(\tau) \rangle &= -\delta^{ab} \delta(\sigma - \tau) \end{aligned} \quad (40)$$

There is no factor of \hbar because we chose to work with operators $\hat{\psi}^a, \hat{\psi}_b$ satisfying $\{\hat{\psi}^a, \hat{\psi}_b\} = \delta_b^a$. Clearly, in this derivation we could instead equally well have started from the second line in (33) and have introduced $\dot{\bar{\psi}}_{k+1/2} = (\bar{\psi}_{k+1} - \bar{\psi}_k)/\epsilon$, $\bar{\psi}_{k+1/2} = (\bar{\psi}_{k+1} + \bar{\psi}_k)/2$, leading again to $\langle \bar{\psi}_{k+1/2} \psi_l \rangle = -\theta_{l,k}$ for the discretized propagators, and therefore also to identical Feynman rules in the continuum limit.

2.3 Weyl-ordering of $N=2$ and $N=1$ Hamiltonians

We will now derive the Weyl-ordered Hamiltonians corresponding to the supersymmetric $N=2$ and $N=1$ Hamiltonians. These are the most interesting fermionic systems, and play the same privileged role as (9) in the bosonic case. As it turns out, different expressions result when we take the independent fermionic fields to have flat or curved indices, and Weyl-order with respect to these independent fields. Let us first consider the $N=2$ quantum Hamiltonian [2]

$$\begin{aligned} H_{N=2} &= \frac{1}{2} g^{-1/4} \pi_i g^{1/2} g^{ij} \pi_j g^{-1/4} - \frac{1}{8} \hbar^2 R_{abcd} \psi_\alpha^a \psi_\alpha^b \psi_\beta^c \psi_\beta^d \\ \pi_i &= p_i - \frac{i\hbar}{2} \omega_{iab} \psi_\alpha^a \psi_\alpha^b \end{aligned} \quad (41)$$

where ω_{iab} is the spin connection and $\alpha = 1, 2$. We can now define Dirac spinors in the following way

$$\psi^a = \frac{1}{\sqrt{2}} (\psi_1^a + i\psi_2^a) \quad ; \quad \bar{\psi}^a = \frac{1}{\sqrt{2}} (\psi_1^a - i\psi_2^a) \quad (42)$$

satisfying the anticommutation relations $\{\bar{\psi}^a, \psi^b\} = \delta^{ab}$. In terms of those, the $N=2$ quantum Hamiltonian corresponds to the following classical action

$$\mathcal{L} = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + \hbar \bar{\psi}^a (\dot{\psi}^a + \dot{x}^j \omega_j^{ab} \psi_b) - \frac{1}{2} \hbar^2 R_{abcd}(\omega) \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d. \quad (43)$$

Its field equations read

$$\frac{\delta}{\delta x^i} \mathcal{L} = -g_{ij} \frac{D}{dt} \dot{x}^j + \hbar R(\omega)_{ijab} \dot{x}^j \bar{\psi}^a \psi^b - \frac{1}{2} \hbar^2 D_i R(\omega)_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d$$

$$\begin{aligned}
& -\omega_{iab} \left(\bar{\psi}^a \frac{\delta}{\delta \bar{\psi}^b} \mathcal{L} + \psi^a \frac{\delta}{\delta \psi^b} \mathcal{L} \right), \\
\frac{\delta}{\delta \psi^a} \mathcal{L} &= \frac{D}{dt} \psi^a - \hbar R(\omega)^a_{bcd} \psi^b \bar{\psi}^c \psi^d, \\
\frac{\delta}{\delta \bar{\psi}^a} \mathcal{L} &= \frac{D}{dt} \bar{\psi}^a - \hbar R(\omega)_{dbc}{}^a \bar{\psi}^d \psi^b \bar{\psi}^c,
\end{aligned} \tag{44}$$

where

$$\frac{D}{dt} \dot{x}^j = \ddot{x}^j + \Gamma_{kl}^j \dot{x}^k \dot{x}^l \tag{45}$$

and

$$\frac{D}{dt} \psi^a = \dot{\psi}^a + \dot{x}^i \omega_i{}^a{}_b \psi^b, \quad \frac{D}{dt} \bar{\psi}^a = \dot{\bar{\psi}}^a + \dot{x}^i \omega_i{}^{ab} \bar{\psi}_b. \tag{46}$$

The invariance of the action under the two rigid supersymmetries follows by contracting the field equations with the variations

$$\begin{aligned}
\delta x^i &= \bar{\epsilon} e_a^i \psi^a + \bar{\psi}^a e_a^i \epsilon \\
\delta \psi^a &= \dot{x}^i e_i^a \epsilon - \delta x^i \omega_i{}^{ab} \psi_b \\
\delta \bar{\psi}^a &= -\bar{\epsilon} e_i^a \dot{x}^i - \delta x^i \omega_i{}^{ab} \bar{\psi}_b.
\end{aligned} \tag{47}$$

All terms then cancel (using the cyclic and the Bianchi identities for $D_i R_{abcd}$). Since this action is in Euclidean time, it is not hermitian, nor is $(\delta \psi^a)^\dagger$ equal to $\delta \bar{\psi}^a$, but it can be obtained from a hermitian action in Minkowski space by the Wick rotation $t_M = -it_E$. The classical Noether charge for supersymmetry reads $e_{ia}(x) \psi_\alpha^a \dot{x}^i$ with $\alpha = 1, 2$, while the quantum charge is given by $Q_\alpha = e_{ia}(x) \psi_\alpha^a g^{1/4} \pi_i g^{-1/4} = g^{-1/4} \pi_i g^{1/4} e_{ia}(x) \psi_\alpha^a$. It is hermitian and Einstein invariant, and $\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta} H$. This shows that the Hamiltonian is supersymmetric and that supersymmetry is preserved at the quantum level. Note that the variation $\delta \psi^a + \delta x^i \omega_i{}^{ab} \psi_b$ is covariant; the ‘pull-back’ term $\delta x^i \omega_i{}^{ab} \psi_b$ is due to the presence of fermionic equation of motion terms in the bosonic equation of motion.

Weyl-ordering of the bosonic part of the Hamiltonian gives rise to the usual contribution (13)

$$\frac{1}{8} \hbar^2 \left(R + g^{ij} \Gamma_{il}^k \Gamma_{jk}^l \right) \tag{48}$$

For the Weyl-ordering with respect to the fermions, we first choose the fermions with flat indices as our independent variables. Clearly the terms quadratic in the fermions yield no contribution, because their anticommutator is proportional to δ^{ab} , which gives zero upon contraction with ω_{iab} . For the terms quartic in the fermions, we use the identity (see e.g. [36])

$$\frac{1}{8} \left\{ [\bar{\psi}^a, \psi^b], [\bar{\psi}^c, \psi^d] \right\} = \left(\bar{\psi}^a \psi^b \bar{\psi}^c \psi^d \right)_W + \frac{1}{4} \delta^{ad} \delta^{bc} \tag{49}$$

Since the four fermion terms in the Hamiltonian have exactly the same symmetry as the operator on the left hand side of the above identity, we can easily deduce the fermionic contribution to the Weyl-ordered Hamiltonian to be

$$-\frac{1}{8}\hbar^2 \left(R + g^{ij} \omega_{ia}{}^b \omega_{jb}{}^a \right) \quad (50)$$

Adding this term to the bosonic contribution (48) we find, for the $N=2$ case,

$$\frac{1}{8}\hbar^2 g^{ij} \left(\Gamma_{il}^k \Gamma_{jk}^l - \omega_{ia}{}^b \omega_{jb}{}^a \right) \quad (51)$$

Alternatively, one can take the fermions with curved indices, namely $\psi^i = \frac{1}{\sqrt{2}}(\psi_1^i + i\psi_2^i)$ and $\bar{\psi}_i = g_{ij} \frac{1}{\sqrt{2}}(\psi_1^j - i\psi_2^j)$, as independent variables, and Weyl-order with respect to these. One finds then that now the bosonic and fermionic contributions exactly cancel, or, in other words, the $N=2$ supersymmetric Hamiltonian expressed in these variables is already Weyl-ordered [36].

The $N=2$ Hamiltonian cannot be interpreted as the regulator of a corresponding quantum field theory, because each of the $2n$ ψ_α^a ($\alpha = 1, 2$) would have to correspond to a Dirac matrix, whereas there are only n Dirac matrices in an n -dimensional quantum field theory (with x^i with $i = 1 \dots n$). However, it plays a role in the path integral evaluation of the index of the $\bar{\partial}$ -operator of the Dolbeault complex [2].

We now consider the $N=1$ supersymmetric case, where only one species of Majorana fermions is present, which makes the generalization of the previous result non-straightforward. The Hamiltonian in this case is equal to

$$\begin{aligned} H_{N=1} &= \frac{1}{2} g^{-1/4} \pi_i g^{1/2} g^{ij} \pi_j g^{-1/4} - \frac{1}{8} \hbar^2 R \\ \pi_i &= p_i - \frac{i\hbar}{2} \omega_{iab} \psi^a \psi^b \end{aligned} \quad (52)$$

where $a = 1 \dots n$, and the fermions satisfy the usual relation $\{\psi^a, \psi^b\} = \delta^{ab}$.

This Hamiltonian cannot be obtained by truncation of the $N=2$ Hamiltonian. For example, putting $\psi_1 - \psi_2 = 0$ requires $\epsilon_1 + \epsilon_2 = 0$, see (47), but the resulting Hamiltonian has $-\frac{1}{16}\hbar^2 R$ instead of $-\frac{1}{8}\hbar^2 R$, and is no longer supersymmetric. The reason is that the truncation $\psi_1 - \psi_2 = 0$ is no longer consistent at the quantum level since $\{\psi_1 - \psi_2, \psi_1 - \psi_2\}$ is nonzero. The easiest way to obtain (52), is to start from the $N=1$ action with $\mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \psi^a \frac{D}{dt} \psi^a$, to construct the Noether quantum supersymmetry charge $Q = g^{1/4} e_a^i \psi^a \pi_i g^{-1/4}$, with $\pi_i = p_i - \frac{i\hbar}{2} \omega_{iab} \psi^a \psi^b$, and then to evaluate $H = \frac{1}{2} \{Q, Q\}$. The algebra is the same as used to evaluate $\{\not{D}, \not{D}\}$ in (111), and leads to (52).

In order to construct coherent states, we cannot work with Dirac brackets or Majorana spinors, but we need creation and annihilation operators. There are two ways to achieve this: either by combining interacting Majorana spinors or by adding free Majorana spinors.

We will first combine these Majorana fermions into complex spinors $\chi, \bar{\chi}$ in the following way

$$\chi^A = \frac{1}{\sqrt{2}}(\psi^{2A-1} + i\psi^{2A}) \quad ; \quad \bar{\chi}^A = \frac{1}{\sqrt{2}}(\psi^{2A-1} - i\psi^{2A}) \quad (53)$$

where $A = 1 \dots n/2$ and $\{\chi^A, \bar{\chi}^B\} = \delta^{AB}$. The inverse relations are given by

$$\begin{aligned} \psi^a &= \frac{1}{\sqrt{2}}(\chi^{(a+1)/2} + \bar{\chi}^{(a+1)/2}) & \text{if } a \text{ odd} \\ \psi^a &= -\frac{i}{\sqrt{2}}(\chi^{a/2} - \bar{\chi}^{a/2}) & \text{if } a \text{ even} \end{aligned} \quad (54)$$

We substitute (54) into (52) and define Weyl-ordering with respect to the operators $\chi, \bar{\chi}$ in the usual way. It is easy to prove, considering separately the cases a, b odd or even, that the following equality holds

$$\psi^a \psi^b = \left(\psi^a \psi^b \right)_W + \frac{1}{2} \delta^{ab} \quad (55)$$

We can now derive the Weyl-ordered expression corresponding to the $N=1$ Hamiltonian. The bosonic part yields the same contribution as before, see (48), and, because of the above identity, the part quadratic in the fermions is again already Weyl-ordered. It remains to consider the term quartic in the fermions $H^{(\text{quartic})} = -\frac{1}{8} \hbar^2 g^{ij} \omega_{iab} \omega_{jcd} \psi^a \psi^b \psi^c \psi^d$. If only one pair of fermions gives rise to a non-trivial anti-commutator, we obtain a contribution proportional to $g^{ij} \omega_{ia}^c \omega_{jcb} \psi^a \psi^b$, which vanishes identically for symmetry reasons. We therefore need also the second pair of fermions to yield a non-vanishing anticommutator in order to find a non-zero contribution. Now there are two possibilities: the two sets of fermions are in different sectors (have different Dirac index A), in which case we only need to Weyl-order both pairs separately using (55), leading to

$$\begin{aligned} g^{ij} \omega_{iab} \omega_{jcd} \psi^a \psi^b \psi^c \psi^d &= \left(g^{ij} \omega_{iab} \omega_{jcd} \psi^a \psi^b \psi^c \psi^d \right)_W \\ &\quad + \frac{1}{2} g^{ij} \omega_{ia}^b \omega_{jb}^a \end{aligned} \quad (56)$$

where we should remember that, if we define $A = [(a+1)/2]$ etc., not all four indices A, B, C, D are identical. The second possibility corresponds to the case A, B, C, D all identical. In that case a, b, c, d are all equal to $2k+1$ or $2k+2$, and we only need

to consider the case that two of them are equal to $2k+1$ and the other two equal to $2k+2$. In this case the Weyl ordered expression vanishes¹⁴. Hence this leads to the same result, so that in fact (56) is valid for all a, b, c, d . For instance $\psi^1\psi^2\psi^2\psi^1 = \frac{1}{4}$ and $(\psi^1\psi^2\psi^2\psi^1)_W = 0$; we end up with a coefficient $\frac{1}{2}$ in (56) because we obtain the same contribution from a term $\psi^1\psi^2\psi^1\psi^2 = -\frac{1}{4}$. Adding the bosonic and fermionic parts, we find the total contribution from Weyl-ordering to the scalar potential

$$\frac{1}{8}\hbar^2 \left(R + g^{ij}\Gamma_{il}^k\Gamma_{jk}^l \right) - \frac{1}{16}\hbar^2 g^{ij}\omega_{ia}{}^b\omega_{jb}{}^a \quad (57)$$

As one might have anticipated, the Weyl-ordering of the Majorana fermions yields half the result for Dirac fermions, see (51).

The other way to construct Dirac fermions from Majorana fermions is to add a second set of free Majorana fermions. Denoting the original fermions ψ^a by ψ_1^a , and the new ones by ψ_2^a , we again construct Dirac fermions χ and $\bar{\chi}$ (as in (42)), but then we use the $N=2$ formulation given before. The four fermion term in the Hamiltonian now reads

$$- \frac{\hbar^2}{8} \psi_1^a \psi_1^b \psi_1^c \psi_1^d \omega_{iab} \omega_{jcd} g^{ij} \quad (58)$$

and we should in this case Weyl order it with respect to χ^a and $\bar{\chi}^a$. Again, (55) holds for ψ_1^a , even though the definition of χ^a and $\bar{\chi}^a$ is now different. Using the operator identity

$$\begin{aligned} \psi_1^a \psi_1^b \psi_1^c \psi_1^d &= \frac{1}{4} (\delta^{ab} \delta^{cd} - \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) \\ &\quad + \left(\frac{1}{2} \delta^{ab} (\psi_1^c \psi_1^d)_W + \text{five more terms} \right) \\ &\quad + (\psi_1^a \psi_1^b \psi_1^c \psi_1^d)_W \end{aligned} \quad (59)$$

we find that the two-fermion terms vanish due to anti-symmetry while the double contractions yield the same answer as in (57).

Finally we can also Weyl-order the $N=1$ Hamiltonian with respect to the fermionic variables χ and $\bar{\chi}$ in (53), but now with curved rather than flat indices. In that case one expects to be left with a remainder

$$\frac{1}{8}\hbar^2 \left(R + \frac{1}{2} g^{ij} \Gamma_{il}^k \Gamma_{jk}^l \right). \quad (60)$$

¹⁴To see this, take $k=0$. Any real linear combination of ψ^1 and ψ^2 can be written as $\alpha\psi + \bar{\alpha}\bar{\psi}$ for some complex α . The Weyl ordered expression of an arbitrary combination of ψ^1 and ψ^2 is proportional to the sum over all graded permutations of $\alpha, \beta, \gamma, \delta$ of the operator $(\alpha\psi + \bar{\alpha}\bar{\psi})(\beta\psi + \bar{\beta}\bar{\psi})(\gamma\psi + \bar{\gamma}\bar{\psi})(\delta\psi + \bar{\delta}\bar{\psi})$. This equals $\alpha\bar{\beta}\gamma\bar{\delta}\psi\bar{\psi} + \bar{\alpha}\beta\bar{\gamma}\delta\bar{\psi}\psi$, and the sum over graded permutations of the ordinary constants $\alpha, \bar{\beta}, \gamma, \bar{\delta}$ and $\bar{\alpha}, \beta, \bar{\gamma}, \delta$ clearly vanishes.

2.4 Evaluation of supersymmetric transition elements

2.4.1 The $N=2$ case

We will first compute the transition element for the $N=2$ case using the path integral formulation. We shall obtain all terms through order β . We rescale $\tau = t/\beta$ to make the β -dependence more explicit and facilitate keeping track of the order in the expansion in β . First note that we can write

$$S = S_{\text{bos}} + S_{\text{fer}}^{\text{kin}} + S_{\text{fer}}^{\text{int}} \quad (61)$$

Here S_{bos} contains all terms with only bosonic or ghost fields. Note however that it is not identical to the action we wrote for the bosonic path integral, as the Weyl-ordering of the fermionic terms also gives a contribution $\sim (R + \omega^2)$. The contributions to the path integral involving only this part of the action can trivially be found from the bosonic case, because the latter terms can, through order β , simply be replaced by their classical expectation values. The other two terms are given by

$$\frac{1}{\hbar} S_{\text{fer}}^{\text{kin}} = \int_{-1}^0 d\tau \delta_{ab} \bar{\psi}^a \dot{\psi}^b - \delta_{ab} \bar{\psi}^a(0) \psi^b(0) \quad (62)$$

and, after integrating out the momenta p_i ,

$$\frac{1}{\hbar} S_{\text{fer}}^{\text{int}} = \int_{-1}^0 d\tau \left[\dot{x}^i \omega_{iab} \bar{\psi}^a \psi^b - \frac{\hbar}{2} \beta R_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d \right] \quad (63)$$

In the background field approach we decompose $\psi(\tau) = \psi_{\text{bg}}(\tau) + \psi_{\text{qu}}(\tau)$, and $\bar{\psi}(\tau) = \bar{\psi}_{\text{bg}}(\tau) + \bar{\psi}_{\text{qu}}(\tau)$. We choose $\bar{\psi}_{\text{bg}}(\tau)$ and $\psi_{\text{bg}}(\tau)$ to be the solutions to the equations of motion of the kinetic part of the action, (62), which satisfy the boundary conditions, i.e. we take $\bar{\psi}_{\text{bg}}^a(\tau) = \bar{\eta}^a$ and $\psi_{\text{bg}}^a(\tau) = \chi^a$. Since we require $\bar{\psi}^a(0) = \bar{\eta}^a$ and $\psi^a(-1) = \chi^a$, this implies that the quantum fields need to satisfy the boundary conditions $\bar{\psi}_{\text{qu}}^a(0) = 0$ and $\psi_{\text{qu}}^a(-1) = 0$. Then (62) simplifies to $\int_{-1}^0 \bar{\psi}_{\text{qu}}^a \dot{\psi}_{\text{qu}}^a d\tau - \bar{\eta}^a \chi^a$, i.e., there are no terms linear in fermionic quantum fields in (62). Of course, (63) does contain such terms.

We can now compute the transition element by expanding $\exp\left(-\frac{1}{\hbar} S_{\text{fer}}^{\text{int}}\right)$ and contracting the quantum fields, using

$$\frac{1}{\hbar} S_{\text{bos}}^{\text{int}} = (29) + \frac{1}{8} \hbar \beta \int_{-1}^0 (-R + g^{ij} \omega_{iab} \omega_j^{ab}) \quad (64)$$

and the propagators for the fermions given in (40). When we expand $\exp\left(-\frac{1}{\hbar} S_{\text{fer}}^{\text{int}}\right)$ we will for the first term in this expansion only need the contraction at equal time

$\langle \bar{\psi}^a(\tau)\psi^b(\tau) \rangle = -\frac{1}{2}\hbar\delta^{ab}$. Hence, the first term in (63) will yield no contribution, whereas the second term contributes $\frac{1}{2}\beta\hbar R_{ab}\bar{\eta}^a\chi^b - \frac{1}{8}\beta\hbar R$. The term $-\frac{1}{8}\beta\hbar R$ cancels the R term in (64). Next consider the term $\frac{1}{2}\left(\frac{1}{\hbar}S_{\text{fer}}^{\text{int}}\right)^2$, where only terms from the square of the $\dot{x}\omega\bar{\psi}\psi$ term contribute to this order. When we contract one pair of fermions, we find the $\omega\omega$ term in (67). When we contract only the four fermionic fields, or two \dot{x} fields and two fermionic fields, one finds zero. Finally, we can contract four fermionic and two bosonic fields. This yields a contribution $-\frac{1}{8}\beta\hbar g^{ij}\omega_{ia}{}^b\omega_{jb}{}^a$, which cancels the $\omega\omega$ term in (64). Contractions involving the other bosonic fields are again of higher order and need not be considered.

Taking all contributions into account, we find for the amplitude

$$\begin{aligned} \langle z, \bar{\eta} | \exp\left(-\frac{\beta}{\hbar}\hat{H}\right) | y, \chi \rangle &= (2\pi\beta\hbar)^{-n/2} \exp\left(-\frac{1}{\hbar}S_B - \frac{1}{\hbar}S_F\right) \\ &\quad \left[1 - \frac{1}{12}\beta\hbar R(z) - \frac{1}{12}R_{ij}(z)(y-z)^i(y-z)^j + \frac{1}{2}\beta\hbar R_{ab}(z)\bar{\eta}^a\chi^b\right] \end{aligned} \quad (65)$$

where

$$\begin{aligned} S_B &= \frac{1}{2\beta}g_{ij}(z)(y-z)^i(y-z)^j + \frac{1}{4\beta}\partial_k g_{ij}(z)(y-z)^i(y-z)^j(y-z)^k \\ &\quad + \frac{1}{12\beta}\left(\partial_k\partial_l g_{ij}(z) - \frac{1}{2}g_{mn}(z)\Gamma_{ij}^m(z)\Gamma_{kl}^n(z)\right)(y-z)^i(y-z)^j(y-z)^k(y-z)^l \end{aligned} \quad (66)$$

is the expansion through order β of the length of the geodesic joining z and y (cf. (11)), and

$$\begin{aligned} S_F &= -\hbar\delta_{ab}\bar{\eta}^a\chi^b - \hbar(y-z)^i\omega_{iab}(z)\bar{\eta}^a\chi^b \\ &\quad - \frac{1}{2}\hbar(y-z)^i(y-z)^j\left(\partial_i\omega_{jab}(z) + \omega_{ia}{}^c(z)\omega_{jcb}(z)\right)\bar{\eta}^a\chi^b \\ &\quad - \frac{1}{2}\beta\hbar^2 R_{abcd}(z)\bar{\eta}^a\chi^b\bar{\eta}^c\chi^d \end{aligned} \quad (67)$$

The ω and $\partial\omega$ terms are obtained by expanding the first term in (63). In the continuum limit, S_F becomes the fermionic action, including the correct boundary term, as derived above (38),

$$S_F = \hbar \int_{-1}^0 d\tau \left(\delta_{ab}\bar{\psi}^a\dot{\psi}^b + \dot{x}^i\omega_{iab}\bar{\psi}^a\psi^b - \frac{1}{2}\beta\hbar R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d \right) - \hbar\delta_{ab}\bar{\eta}^a\psi_{\text{bg}}^b(0). \quad (68)$$

We can easily check that the expansion through order β of (68) indeed equals the expression in (67) when the equations of motion are imposed. The latter follow from

(44) (of course we also need the bosonic part of the action to find the full equations of motion)

$$\begin{aligned}
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k - \hbar R^i_{jab} \dot{x}^j \bar{\psi}^a \psi^b + \frac{1}{2} \hbar^2 g^{ij} (\partial_j R_{abcd}) \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d &= 0 \\
\dot{\psi}^a + \dot{x}^i \omega_i^a{}_b \psi^b - \hbar R^a_{bcd} \psi^b \bar{\psi}^c \psi^d &= 0 \\
\dot{\bar{\psi}}^a + \dot{x}^i \omega_i^a{}_b \bar{\psi}^b - \hbar R^a_{bcd} \bar{\psi}^b \bar{\psi}^c \psi^d &= 0
\end{aligned} \tag{69}$$

where a dot denotes differentiation with respect to t . We can now expand the Lagrangian in a Taylor series around its value at $t = 0$, and then do the trivial time integrations. This yields

$$S = \beta L(0) - \frac{1}{2} \beta^2 \dot{L}(0) + \dots \tag{70}$$

We thus expand all fields in the Lagrangian around their values at $t = 0$, making use of the equations of motion (69). The expansions up to the order we need are given by

$$\begin{aligned}
\dot{x}^i(0) &= \frac{(z-y)^i}{\beta} + \frac{\beta \hbar}{2} (z-y)^j R^i_{jab} \bar{\eta}^a \chi^b \\
\bar{\psi}^a(0) &= \bar{\eta}^a \\
\psi^a(0) &= \chi^a - (z-y)^i \omega_i^a{}_b \chi^b \\
\dot{\psi}^a(0) &= \frac{\psi^a(0) - \chi^a}{\beta} - \frac{1}{2} \frac{(z-y)^i (z-y)^j}{\beta} (\partial_i \omega_j^a{}_b - \omega_i^a{}_c \omega_j^c{}_b) \chi^b
\end{aligned} \tag{71}$$

Inserting these expansions into S_F in (68) and (70) yields the expression (67).

We will now show that the final result for the transition amplitude can again be written as the product of three factors: a term containing only the scalar curvature which is related to the trace anomaly, the exponent of the classical action, and the square root of, in this case, the supersymmetric generalization of the Van Vleck determinant. The latter we define by

$$D_S = \text{sdet} D_{AB} \quad ; \quad D_{AB} \equiv -\frac{\partial}{\partial \Phi^A} (S_B + S_F) \frac{\overleftarrow{\partial}}{\partial \Phi^B} \tag{72}$$

where $\Phi^A = (z^i, \bar{\eta}^a)$ and $\Phi^B = (y^j, \chi^b)$, and for S_B and S_F we substitute the expressions (66) and (67). To evaluate D_S write

$$D_{AB} = \begin{pmatrix} A_{ij} & B_{ib} \\ C_{aj} & D_{ab} \end{pmatrix} \tag{73}$$

We find, expanding in normal co-ordinates around z to simplify the expressions,

$$\begin{aligned}
A_{ij} &= \frac{1}{\beta} g_{ij}(z) + \hbar \partial_i \omega_{jab}(z) \bar{\eta}^a \chi^b \\
&\quad - \frac{\hbar}{2} \left(\partial_i \omega_{jab}(z) + \partial_j \omega_{iab}(z) + \omega_{ia}{}^c(z) \omega_{jcb}(z) + \omega_{ja}{}^c(z) \omega_{icb}(z) \right) \bar{\eta}^a \chi^b \\
B_{ib} &= -\hbar \omega_{iab}(z) \bar{\eta}^a \\
C_{aj} &= \hbar \omega_{jab}(z) \chi^b \\
D_{ab} &= \hbar \delta_{ab} + \hbar (y-z)^i \omega_{iab}(z) + \frac{\hbar}{2} (y-z)^i (y-z)^j \left(\partial_i \omega_{jab}(z) + \omega_{ia}{}^c(z) \omega_{jcb}(z) \right) \\
&\quad + \hbar \beta \left(R_{abcd}(z) - R_{adcb}(z) \right) \bar{\eta}^c \chi^d
\end{aligned} \tag{74}$$

We do not need terms of order β in B and C , since $D_S = \det A \det^{-1}(D - CA^{-1}B)$ and A^{-1} is already of order β . Writing $A_{ij} = \frac{1}{\beta} g_{ik}(\delta^k_j + \beta \hbar a^k_j)$ and $D_{ab} = \hbar(\delta_{ab} + d_{ab})$, we can write the expansion of the super Van Vleck determinant as

$$D_S^{1/2} = (\beta \hbar)^{-n/2} g^{1/2}(z) \left[1 + \frac{1}{2} \beta \hbar \text{tra} + \frac{1}{2} \beta \hbar \text{tr} C B - \frac{1}{2} \text{tr} d + \frac{1}{8} (\text{tr} d)^2 + \frac{1}{4} \text{tr}(d^2) \right] \tag{75}$$

Multiplying by $g^{-1/4}(z) g^{-1/4}(y)$ to transform $D_S^{1/2}$ into a bi-scalar, we obtain

$$\begin{aligned}
\tilde{D}_S^{1/2} &\equiv (\beta \hbar)^{n/2} g^{-1/4}(z) D_S^{1/2} g^{-1/4}(y) \\
&= \left[1 - \frac{1}{12} R_{ij}(z) (y-z)^i (y-z)^j + \frac{1}{2} \beta \hbar R_{ab} \bar{\eta}^a \chi^b \right]
\end{aligned} \tag{76}$$

So indeed we can write

$$\langle z, \bar{\eta} | \exp \left(-\frac{\beta}{\hbar} \hat{H} \right) | y, \chi \rangle = (2\pi \hbar \beta)^{-n/2} \tilde{D}_S^{1/2} \exp \left(-\frac{1}{\hbar} (S_B + S_F) \right) \left[1 - \frac{1}{12} \beta \hbar R \right] \tag{77}$$

All terms involving the Ricci curvature in (65) are thus completely accounted for by the super Van Vleck determinant, which clearly would not have been the case if we had used the ordinary determinant. Finally we note that if we would have rescaled all fermions by a factor of $(\beta \hbar)^{-1/2}$, then all classical terms are proportional to $1/(\beta \hbar)$, while all one-loop terms in (76) are $\beta \hbar$ independent and the two-loop term in (77) remains proportional to $\beta \hbar$ [24].

2.4.2 The $N=1$ case

Next we consider the $N=1$ case. We will first evaluate the transition element when we double the number of Majorana fermions, and afterwards consider the case that we combine the Majorana fermions into half as many Dirac fermions. In the first case we add n free fermions ψ_2^a and combining $1/\sqrt{2}(\psi_1^a + i\psi_2^a) = \psi^a$ and $1/\sqrt{2}(\psi_1^a - i\psi_2^a) = \bar{\psi}^a$

we construct the path integral with the corresponding coherent states. The kinetic part of the fermionic action is again

$$\frac{1}{\hbar} S_{\text{fer}}^{\text{kin}} = \int_{-1}^0 d\tau \delta_{ab} \bar{\psi}^a \dot{\psi}^b - \delta_{ab} \bar{\psi}^a(0) \psi^b(0) \quad (78)$$

and yields the free field equations $\dot{\psi}^a = \dot{\bar{\psi}}^a = 0$, but the interaction part containing the fermions is now equal to

$$\frac{1}{\hbar} S_{\text{fer}}^{\text{int}} = \int_{-1}^0 d\tau \left[\frac{1}{2} \dot{x}^i \omega_{iab} \psi_1^a \psi_1^b \right] - \bar{\eta} \chi \quad (79)$$

Note that the terms linear in $\psi_{\text{qu}}(0)$ in (78) cancel.

We have introduced an extra set of fermions that do not couple to any of the other fields, this way making certain that we do not alter the dynamics. In order to preserve local Lorentz invariance we require that the fermions ψ_2^a are inert under local Lorentz transformations. The extra contribution to the scalar potential from Weyl-ordering the $N=1$ Hamiltonian is of the form $\frac{\hbar^2}{8}(\Gamma\Gamma + R) - \frac{1}{16}\omega\omega$ to which one should add the term $-\frac{\hbar^2}{8}R$ from (52), and its integral can be replaced to order β by β times its classical value. The purely bosonic sector of the path integral can be evaluated exactly as before, so we only need to consider the sector involving fermions. We use a background field expansion as in the $N=2$ case, again with constant background fields which satisfy the boundary conditions, and substitute $\psi_1^a = \frac{1}{\sqrt{2}}(\bar{\eta}^a + \chi^a) + \psi_{1,\text{qu}}^a$ into (79). Since the interactions depend only on ψ_1 , it will be easier to use the propagator for $\psi_{1,\text{qu}}$

$$\langle \psi_{1,\text{qu}}^a(\sigma) \psi_{1,\text{qu}}^b(\tau) \rangle = \frac{1}{2} \delta^{ab} (\theta(\sigma - \tau) - \theta(\tau - \sigma)) \quad (80)$$

which can trivially be found from the $\bar{\psi}\psi$ propagator in (40) and the propagators $\langle \psi\psi \rangle = \langle \bar{\psi}\bar{\psi} \rangle = 0$. We now evaluate $\langle \exp -\frac{1}{\hbar} S_{\text{fer}}^{\text{int}} \rangle$. We first consider terms from contractions in $-\frac{1}{\hbar} S_{\text{fer}}^{\text{int}}$. The equal time contraction of $\psi_1^a \psi_1^b$ in (80) vanishes, so this term will yield no contribution. Also the contribution from the contraction in $\dot{x}^i \omega_{iab}$ in (79) vanishes, as it is proportional to $\int_{-1}^0 d\tau \langle \dot{x}^i(\tau) x^j(\tau) \rangle$ which is zero. Next we consider the term $\frac{1}{2} \left(\frac{1}{\hbar} S_{\text{fer}}^{\text{int}} \right)^2$. The contraction of $\dot{x}^i(\sigma)$ with $\dot{x}^j(\tau)$ is of order β , and would contribute, but its integral over σ and τ vanishes. One $\dot{x}^i(\sigma)$ contracted with $x^j(\tau)$ is also of order β , but the other \dot{x}^j would leave a factor \dot{x}_{bg}^j which is of order $\beta^{1/2}$, so this term is of higher order. The contraction of only one pair of fermions vanishes for symmetry reasons, so there is now no $\omega\omega$ in (82). The contraction of all four fermions is nonvanishing, and produces the term $\frac{1}{16}(z^i - y^i)(z^j - y^j)\omega_{ia}^b \omega_{jb}^a$ in (81). Finally, we can contract four fermionic fields and two bosonic fields \dot{x}^i . This

yields $-\frac{1}{16}\beta\hbar g^{ij}\omega_{ia}{}^b\omega_{jb}{}^a$, and this term exactly cancels a similar noncovariant term in the scalar potential due to Weyl-ordering. Adding all contributions we find the transition element for the $N=1$ case with fermion doubling

$$\begin{aligned} \langle z, \bar{\eta} | \exp\left(-\frac{\beta}{\hbar}\hat{H}\right) | y, \chi \rangle &= (2\pi\beta\hbar)^{-n/2} \exp\left(-\frac{1}{\hbar}(S_B + S_F)\right) \\ &\quad \left[1 + \frac{1}{24}\beta\hbar R(z) - \frac{1}{12}R_{ij}(z)(y-z)^i(y-z)^j \right. \\ &\quad \left. + \frac{1}{16}(y-z)^i(y-z)^j\omega_{ia}{}^b(z)\omega_{jb}{}^a(z) \right] \end{aligned} \quad (81)$$

where

$$\begin{aligned} \frac{1}{\hbar}S_F &= -\delta_{ab}\bar{\eta}^a\chi^b - \frac{1}{4}(y-z)^i\omega_{iab}(z)(\bar{\eta}^a + \chi^a)(\bar{\eta}^b + \chi^b) \\ &\quad - \frac{1}{8}(y-z)^i(y-z)^j\partial_i\omega_{jab}(z)(\bar{\eta}^a + \chi^a)(\bar{\eta}^b + \chi^b) \end{aligned} \quad (82)$$

and S_B is the bosonic part of the classical action. It is the same result as obtained directly from operator methods [24].

The terms in S_F are obtained by expanding the following classical continuum action around z

$$\begin{aligned} \frac{1}{\hbar}S &= -\bar{\eta}^a\psi^a(0) + \int_{-1}^0 dt \left[\frac{1}{\beta\hbar} \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \bar{\psi}^a \dot{\psi}^a \right. \\ &\quad \left. + \frac{1}{2} \dot{x}^i \omega_{iab} \psi_1^a \psi_1^b \right]. \end{aligned} \quad (83)$$

The equations of motion read

$$\begin{aligned} 0 &= \ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k - \beta\hbar \frac{1}{4} \dot{x}^j R_{jab}^i (\psi + \bar{\psi})^a (\psi + \bar{\psi})^b \\ 0 &= \dot{\psi}^a + \frac{1}{2} \dot{x}^i \omega_i{}^{ab} (\psi + \bar{\psi})^b \\ 0 &= \dot{\bar{\psi}}^a + \frac{1}{2} \dot{x}^i \omega_i{}^{ab} (\psi + \bar{\psi})^b \end{aligned} \quad (84)$$

where $R_{ijab} = \partial_i\omega_{jab} + \omega_{iac}\omega_{jcb} - (i \leftrightarrow j)$. From them one derives further

$$\begin{aligned} \dot{x}^i(0) &= (z-y)^i - \frac{1}{2}\Gamma_{jk}^i(z)(z-y)^j(z-y)^k + \frac{\beta\hbar}{8}(z-y)^j R_{jab}^i (\bar{\eta} + \chi)^a (\bar{\eta} + \chi)^b + \dots \\ \psi^a(0) &= \chi^a - \frac{1}{2}(z-y)^k \omega_k{}^{ab} (\bar{\eta} + \chi)^b + \frac{1}{4}(z-y)^i(z-y)^j \partial_i\omega_j{}^{ab} (\bar{\eta} + \chi)^b + \dots \end{aligned} \quad (85)$$

Substituting these results into $S = -\bar{\eta}^a\psi^a(0) + L(0) - \frac{1}{2}\frac{d}{d\tau}L(0) + \dots$, the contribution from $-\frac{1}{2}\frac{d}{d\tau}(\bar{\psi}\dot{\psi})$ cancels the terms in $-\bar{\eta}^a\psi^a(0) + \bar{\psi}^a(0)\dot{\psi}^a(0)$, and one indeed arrives at (82).

We will now show that the expression for the propagator can again be written as the product of the super Van Vleck determinant, the exponent of the classical action, and a term involving the scalar curvature which, as shown in section 3.3, determines the trace anomaly of a spin- $\frac{1}{2}$ field. Defining the super Van Vleck determinant as in the $N=2$ case, we find from (82)

$$\begin{aligned}
A_{ij} &= \frac{1}{\beta} g_{ij}(z) + \frac{\hbar}{8} (\partial_i \omega_{jab}(z) - \partial_j \omega_{iab}(z)) (\bar{\eta}^a + \chi^a) (\bar{\eta}^b + \chi^b) \\
B_{ib} &= -\frac{\hbar}{2} \omega_{iab}(z) (\bar{\eta}^a + \chi^a) \\
C_{aj} &= \frac{\hbar}{2} \omega_{jab}(z) (\bar{\eta}^b + \chi^b) \\
D_{ab} &= \hbar \delta_{ab} + \frac{\hbar}{2} (y-z)^i \omega_{iab}(z) + \frac{\hbar}{4} (y-z)^i (y-z)^j \partial_i \omega_{jab}(z).
\end{aligned} \tag{86}$$

Using again (75), one finds

$$\tilde{D}_S^{1/2} = \left[1 - \frac{1}{12} R_{ij}(z) (y-z)^i (y-z)^j + \frac{1}{16} (y-z)^i (y-z)^j \omega_{ia}{}^b(z) \omega_{jb}{}^a(z) \right] \tag{87}$$

where only the last term in (75) did contribute and yields the last term in (87). So we can indeed write

$$\langle z, \bar{\eta} | \exp \left(-\frac{\beta}{\hbar} \hat{H} \right) | y, \chi \rangle = (2\pi\hbar\beta)^{-n/2} \tilde{D}_S^{1/2} \exp \left(-\frac{1}{\hbar} (S_B + S_F) \right) \left[1 + \frac{1}{24} \beta \hbar R \right] \tag{88}$$

similarly to the bosonic and $N=2$ supersymmetric case.

We will now repeat the analysis for the $N=1$ case when we do not introduce an extra set of fermions, but instead combine the n Majorana fermions ψ^a into $n/2$ Dirac fermions $\Psi^A, \bar{\Psi}^A$ as $\Psi^A = \frac{1}{\sqrt{2}}(\psi^{2A-1} + i\psi^{2A})$, $\bar{\Psi}^A = \frac{1}{\sqrt{2}}(\psi^{2A-1} - i\psi^{2A})$. The kinetic part of the fermionic action is now equal to

$$\frac{1}{\hbar} S_{\text{fer}}^{\text{kin}} = \int_{-1}^0 d\tau \delta_{AB} \bar{\Psi}^A \dot{\Psi}^B - \delta_{AB} \bar{\Psi}^A(0) \Psi^B(0) \tag{89}$$

The interaction part containing the fermions is still equal to

$$\frac{1}{\hbar} S_{\text{fer}}^{\text{int}} = \int_{-1}^0 d\tau \left[\frac{1}{2} \dot{x}^i \omega_{iab} \psi^a \psi^b \right] \tag{90}$$

but now the ψ^a should be expressed in terms of Ψ^A and $\bar{\Psi}^A$. We again make a background field decomposition as $\Psi^A = \chi^A + \Psi_{\text{qu}}^A$, $\bar{\Psi}^a = \bar{\eta}^A + \bar{\Psi}_{\text{qu}}^A$. Again it will be convenient to rewrite the propagators for the Dirac fermions in terms of the Majorana fermions. We now find

$$\langle \psi^a(\sigma) \psi^b(\tau) \rangle = \frac{1}{2} \delta^{ab} (\theta(\sigma - \tau) - \theta(\tau - \sigma)) + K^{ab} \tag{91}$$

where $K^{ab} = \frac{i}{4}\delta^{a+1,b}(1 - (-1)^a) - (a \leftrightarrow b)$ (in other words, K is the matrix $-\frac{1}{2}\tau_2$ in the 2×2 subspaces). Furthermore, we define the ‘background’ Majorana fermions

$$\begin{aligned}\tilde{\psi}^a &= \frac{1}{\sqrt{2}}(\chi^{(a+1)/2} + \bar{\eta}^{(a+1)/2}), & a \text{ odd} \\ \tilde{\psi}^a &= -\frac{i}{\sqrt{2}}(\chi^{a/2} - \bar{\eta}^{a/2}), & a \text{ even}\end{aligned}\quad (92)$$

We start again by considering contractions in $-\frac{1}{\hbar}S_{\text{fer}}^{\text{int}}$. The equal time contraction of $\psi^a\psi^b$ is now equal to K^{ab} . These terms therefore contribute

$$\frac{1}{2}(y-z)^i\omega_{iab}K^{ab} + \frac{1}{4}(y-z)^i(y-z)^j\partial_i\omega_{jab}K^{ab}\quad (93)$$

Next we consider contractions in $\frac{1}{2}\left(\frac{1}{\hbar}S_{\text{fer}}^{\text{int}}\right)^2$. When we contract one pair of fermions, we obtain the tree graph

$$\frac{1}{8}(y-z)^i(y-z)^j\omega_{iab}\omega_{jcd}\left(-4K^{ac}\tilde{\psi}^b\tilde{\psi}^d\right).\quad (94)$$

When we contract two pairs of fermions, we find the one-loop graph

$$\frac{1}{8}(y-z)^i(y-z)^j\omega_{iab}\omega_{jcd}\left(\frac{1}{2}\delta^{bc}\delta^{ad} - 2K^{ac}K^{bd} + K^{ab}K^{cd}\right)\quad (95)$$

The last term is a product of two one-loop graphs, but should of course not be counted as a new two-loop graph. When we contract the two \dot{x} fields, we only find a contribution if in addition we contract all fermionic fields. This yields $-\frac{1}{16}\beta\hbar g^{ij}\omega_{ia}{}^b\omega_{jb}{}^a$, which cancels again a similar contribution from Weyl-ordering (see (57)). Adding all terms, we find for the transition element for the $N=1$ case without fermion doubling

$$\langle z, \bar{\eta} | \exp\left(-\frac{\beta}{\hbar}\hat{H}\right) | y, \chi \rangle = (2\pi\beta\hbar)^{-n/2} E \exp\left(-\frac{1}{\hbar}(S_B + S_F)\right) \left[1 + \frac{1}{24}\beta\hbar R\right]\quad (96)$$

where S_F is equal to the background part of the fermionic action (together with the boundary term) plus the tree graph of (94)

$$\begin{aligned}\frac{1}{\hbar}S_F &= -\delta_{AB}\bar{\eta}^A\chi^B - \frac{1}{2}(y-z)^i\omega_{iab}(z)\tilde{\psi}^a\tilde{\psi}^b \\ &\quad - \frac{1}{4}(y-z)^i(y-z)^j\partial_i\omega_{jab}(z)\tilde{\psi}^a\tilde{\psi}^b \\ &\quad + \frac{1}{2}(y-z)^i(y-z)^j\omega_{iab}\omega_{jcd}K^{ac}\tilde{\psi}^b\tilde{\psi}^d\end{aligned}\quad (97)$$

The one but last term is due to expanding $\omega(z + (z-y)\tau)$ around z . As the notation indicates, to order β S_F is equal to the classical action (89) and (90) with the equations of motion satisfied. Their solution is, however, quite a bit more complicated than it

was in the previous cases, due to the different boundary conditions we have to impose here. All one-loop contributions are contained in E ,

$$\begin{aligned}
E = & \left[1 - \frac{1}{12} R_{ij}(z)(y-z)^i(y-z)^j \right. \\
& + \left(\frac{1}{2}(y-z)^i \omega_{iab} + \frac{1}{4}(y-z)^i(y-z)^j \partial_i \omega_{jab} \right) K^{ab} \\
& + \frac{1}{8}(y-z)^i(y-z)^j \omega_{iab} \omega_{jcd} \left(\frac{1}{2} \delta^{bc} \delta^{ad} \right. \\
& \left. \left. + K^{ab} K^{cd} - 2K^{ac} K^{bd} \right) \right] \quad (98)
\end{aligned}$$

Comparing with (82), (87) and (88) which yield the transition element with fermion doubling we note three differences: (i) the background value of the ψ in the interactions are defined differently, namely $\chi^a + \bar{\eta}^a = \sqrt{2} \tilde{\psi}_1^a$ in (82) and $\tilde{\psi}^a$ in (92), (ii) the boundary term $\bar{\eta} \chi$ contains half as many terms in (97) as in (82), and (iii) there are extra terms in (97) and (98) proportional to K and KK . Yet, as we shall see, these different transition elements yield the same anomalies.

Motivated by our earlier results, we will now compare the expression E to the square root of the super Van Vleck determinant. Defining the super Van Vleck determinant as before, we find using (97)

$$\begin{aligned}
A_{ij} &= \frac{1}{\beta} g_{ij}(z) + \frac{1}{4} (\partial_i \omega_{jab}(z) - \partial_j \omega_{iab}(z)) \tilde{\psi}^a \tilde{\psi}^b \\
&+ \frac{1}{2} (\omega_{iab} \omega_{jcd} + \omega_{jab} \omega_{icd}) K^{ac} \tilde{\psi}^b \tilde{\psi}^d \\
B_{iB} &= -\frac{1}{2} \omega_{iab}(z) (\tilde{\psi}^a \tilde{\psi}^b) \frac{\overleftarrow{\partial}}{\partial \chi^B} \\
C_{Aj} &= \frac{1}{2} \omega_{jab}(z) \frac{\partial}{\partial \bar{\eta}^A} (\tilde{\psi}^a \tilde{\psi}^b) \\
D_{AB} &= \delta_{AB} + \left[\frac{1}{2} (y-z)^i \omega_{iab}(z) + \frac{1}{4} (y-z)^i (y-z)^j \partial_i \omega_{jab}(z) \right. \\
&\quad \left. - \frac{1}{2} \omega_{iac} \omega_{jbd} (z-y)^i (z-y)^j K^{ac} \right] \frac{\partial}{\partial \bar{\eta}^A} (\tilde{\psi}^a \tilde{\psi}^b) \frac{\overleftarrow{\partial}}{\partial \chi^B} \quad (99)
\end{aligned}$$

The expansion of the super Van Vleck determinant through order β can again be written as

$$\begin{aligned}
\tilde{D}_S^{1/2} &= g^{1/4}(z) g^{-1/4}(y) \left[1 + \frac{1}{2} \beta \text{tra} + \frac{1}{2} \beta \text{tr} C B \right. \\
&\quad \left. - \frac{1}{2} \text{tr} d + \frac{1}{8} (\text{tr} d)^2 + \frac{1}{4} \text{tr}(d^2) \right] \quad (100)
\end{aligned}$$

Now using identities such as

$$\delta^{AB} \frac{\partial}{\partial \bar{\eta}^A} (\tilde{\psi}^a \tilde{\psi}^b) \frac{\overleftarrow{\partial}}{\partial \chi^B} = -2K^{ab} \quad (101)$$

and

$$\frac{\partial}{\partial \bar{\eta}^A} (\tilde{\psi}^a \tilde{\psi}^b) \frac{\overleftarrow{\partial}}{\partial \chi^C} \frac{\partial}{\partial \bar{\eta}^C} (\tilde{\psi}^c \tilde{\psi}^d) \frac{\overleftarrow{\partial}}{\partial \chi^A} = 2(K^{ad} K^{bc} - K^{ac} K^{bd}) + \frac{1}{2}(\delta^{ad} \delta^{bc} - \delta^{ac} \delta^{bd}) \quad (102)$$

we find that the super Van Vleck determinant indeed equals E . Hence the same factorization which we found in the case of fermion doubling in (88) also holds in the case of fermion halving, even though the separate factors $\tilde{D}_S^{1/2}$ and S_F are different.

3 Anomaly calculations

In this section we compute the chiral anomalies due to spin-1/2 fermions coupled to external gravity and external Yang-Mills fields. Then we consider the trace anomaly. In appendix A.3 we discuss gravitational anomalies for spin-1/2 and spin-3/2 fields.

In a quantum field theory, the anomalies can be written as the trace over the product of a Jacobian and a regulator. Both the Jacobian and the regulator depend on which fields one considers as the independent fields, for example, for a scalar field ϕ possible choices are ϕ itself and $\tilde{\phi} = \phi g^{1/4}$. Of course, the anomaly itself should not depend on this choice of basis. It has become common practice to take $\tilde{\phi}$ for scalars, and $\tilde{\chi}^\alpha = \chi^\alpha g^{1/4}$ for spinors as basic variables, because then (with the corresponding regulator) the absence of Einstein anomalies becomes obvious. However, in the non-linear sigma models, one uses another regulator. Namely, since we have taken the inner product between x -eigenstates as $\langle x|y \rangle = g(x)^{-1/2} \delta(x-y)$, the momentum operator is represented by $p_i = -\hbar i g^{-1/4} \partial_i g^{1/4}$, and this representation is clearly obtained from $p_i = -\hbar i \partial_i$ by a similarity transformation with $g^{-1/4}(x)$. As a result, the regulator used in non-linear sigma models is no longer $g^{-1/4} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu g^{-1/4}$ but rather $g^{-1/2} \partial_\mu g^{1/2} g^{\mu\nu} \partial_\nu$. In terms of momenta this reads $g^{-1/4} p_i g^{1/2} g^{ij} p_j g^{-1/4}$. We now explain in more detail how the Einstein anomaly vanishes if we take $\tilde{\phi}$ as an independent field.

Given a set of symmetries one wants to preserve at the quantum level in field theories, there exists a method to construct a regulator which yields consistent anomalies and which preserves these symmetries [18]. The basic idea is to construct a mass term which is bilinear in fields and which separately preserves the symmetries. For scalars, an Einstein invariant mass term is clearly $M \tilde{\phi} \tilde{\phi}$ with $\tilde{\phi} = g^{1/4} \phi$. The regulator is then the kinetic operator for these fields, i.e., $\tilde{R} = g^{-1/4} \partial_\mu g^{1/2} g^{\mu\nu} \partial_\nu g^{-1/4}$. The Jacobian for Einstein transformations reads

$$J(\tilde{\phi}) = 1 + \xi^\mu \partial_\mu + \frac{1}{2}(\partial_\mu \xi^\mu) = 1 + \frac{1}{2}(\xi^\mu \partial_\mu + \partial_\mu \xi^\mu) \quad (103)$$

and using an orthonormal basis $\tilde{\phi}_N(x)$ satisfying $\int \tilde{\phi}_M^*(x) \tilde{\phi}_N(x) dx = \delta_{MN}$ the anomaly becomes

$$\begin{aligned}\mathcal{A}_n &= \frac{1}{2} \int \tilde{\phi}_N^*(x) (\xi^\mu \partial_\mu + \partial_\mu \xi^\mu) \tilde{\phi}_N(x) dx \\ &= \frac{1}{2} \int \tilde{\phi}_N^*(x) \xi^\mu \partial_\mu \tilde{\phi}_N(x) dx - \frac{1}{2} \int \partial_\mu \tilde{\phi}_N^*(x) \xi^\mu \tilde{\phi}_N(x) dx.\end{aligned}\quad (104)$$

It is obvious that this vanishes if the basis $\{\tilde{\phi}_N^*\}$ is in one-to-one correspondence with the basis $\{\tilde{\phi}_N\}$, as is the case for e.g. plane waves. In a more invariant language, the vanishing of the Einstein anomaly follows from the fact that the Jacobian is real and anti-hermitian, and the regulator is real and hermitian. The trace of the product of two such operators always vanishes, which can be deduced from the fact that the trace of any real operator (i.e. an operator that satisfies $A(\phi^*) = A(\phi)^*$) is equal to the trace of its hermitian conjugate.

However, as explained above, in order to evaluate anomalies by using equivalent non-linear sigma models, the basis $\tilde{\phi}$ is not directly compatible with the conventions for $\langle x|y\rangle$ we have chosen. For chiral anomalies or trace anomalies, it does not matter whether one uses $\tilde{\phi}$ or ϕ , because a similarity transformation with $g^{1/4}$ has no effect on γ_5 or $\mathbf{1}$: $g^{1/4} \gamma_5 g^{-1/4} = \gamma_5$ and $g^{1/4} \mathbf{1} g^{-1/4} = \mathbf{1}$. But for gravitational anomalies, the basis ϕ is more convenient. As follows from appendix A.3, one finds for Einstein anomalies (= gravitational anomalies for $\tilde{\phi}$) in a covariant notation

$$J = \xi^\mu D_\mu + \frac{1}{2} (D_\mu \xi^\mu) \quad (105)$$

where $D_\mu = \partial_\mu - \frac{1}{2} \Gamma_{\mu\nu}^\nu$ and $(D_\mu \xi^\mu) = \partial_\mu \xi^\mu + \Gamma_{\mu}^\mu{}_\nu \xi^\nu$. The $\Gamma_{\mu\nu}^\nu$ terms cancel in J . The regulator is

$$R = g^{-1/4} \partial_\mu g^{1/2} g^{\mu\nu} \partial_\nu g^{-1/4} = g^{1/4} R_{cov} g^{-1/4} \quad (106)$$

with $R_{cov} = g^{-1/2} \partial_\mu g^{1/2} g^{\mu\nu} \partial_\nu$ the usual covariant scalar D'Alembertian. Using cyclicity of the trace, or equivalently, making a similarity transformation (change of basis), the Einstein anomaly becomes

$$\begin{aligned}\mathcal{A}_n(\text{Ein}) &= \text{Tr} \left(J \exp(R/M^2) \right) \\ &= \frac{1}{2} \text{Tr} \left(g^{-1/4} (\xi^\mu D_\mu + D_\mu \xi^\mu) g^{1/4} \exp(R_{cov}/M^2) \right)\end{aligned}\quad (107)$$

A direct evaluation of this trace using plane waves is given in [37]. In the non-linear sigma model, $\frac{\hbar}{i} \partial_\mu$ becomes $g^{1/4} p_i g^{-1/4}$, and thus

$$\mathcal{A}_n = \frac{1}{2} \text{Tr} (\xi^i \hat{p}_i + \hat{p}_i \xi^i) \exp(-\beta \hat{H}/\hbar) \quad (108)$$

with $\hat{H} = g^{-1/4} p_i g^{1/2} g^{ij} p_j g^{-1/4}$. Note that J comes out Weyl ordered, so that $\mathcal{A}_n(\text{Ein})$ can be evaluated using the methods of the preceding sections. This explains why \hat{H} is the regulator. (Of course, the Einstein anomaly vanishes, as we already explained, but other anomalies can be calculated with the same regulator.)

For spin-1/2 fields, the choice $\tilde{\chi}^\alpha = \chi^\alpha g^{1/4}$ as basic field leads to the same Einstein Jacobian, the field operator is now $g^{1/4} \not{D} g^{-1/4}$ (where \not{D} is the usual covariant Dirac operator), and, as explained in [18], the consistent regulator which leads to vanishing Einstein anomalies is now the square, namely $\tilde{R} = g^{1/4} \not{D} \not{D} g^{-1/4}$. Gravitational anomalies are really local Lorentz anomalies, but by taking a suitable linear combination of Einstein and local Lorentz transformations one obtains covariantly looking expressions. For chiral spin-1/2 fields, one obtains then (see appendix A.3)

$$\mathcal{A}_n(\text{grav}, \text{spin-1/2}) = -\frac{1}{2} \text{Tr}(\xi^\mu D_\mu + D_\mu \xi^\mu) \exp(\tilde{R}/M^2) \quad (109)$$

where $D_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b - \frac{1}{2} \Gamma_{\mu\nu}{}^\nu$, but the $\Gamma_{\mu\nu}{}^\nu$ terms in the Jacobian cancel again. Making the similarity transformation with $g^{1/4}$ one finds (see (111))

$$\mathcal{A}_n(\text{grav}, \text{spin-1/2}) = -\frac{1}{2} \text{Tr}(\xi^i \pi_i + \pi_i \xi^i) \exp(-\beta \hat{H}/\hbar) \quad (110)$$

where $\pi_i = p_i - \frac{1}{2} i \hbar \omega_{iab} \psi^a \psi^b$ and \hat{H} given in (52). This explains why we took this particular quantum Hamiltonian in the non-linear sigma models.

Finally, for spin-3/2 similar results hold. However, here the Jacobian is more complicated. In [2] an expression is given which does not correspond to a linear combination of local symmetries of supergravity. However, as we explain in appendix A.3, the difference (which would be difficult to evaluate for non-linear sigma models) vanishes if one uses as regulator the same regulator $\not{D} \not{D}$ as for spin-1/2. Why this is the correct regulator is also explained in appendix A.3.

3.1 Chiral anomalies in gravitational couplings

The simplest anomaly one can calculate by means of the path integral methods we have developed, is the γ_5 anomaly due to a loop of a spin-1/2 field coupled to external gravitational fields. As regulator for spin-1/2 fermions we take $\not{D} \not{D}$, which can be rewritten as the sum of a d'Alembertian and a gravitational curvature term (Weitzenbock identity)

$$\not{D} \not{D} = D^i D_i + \frac{1}{2} \gamma^i \gamma^j [D_i, D_j]$$

$$\begin{aligned}
&= \frac{1}{\sqrt{g}} D_i^{(0)} \sqrt{g} g^{ij} D_j^{(0)} + \frac{1}{8} \gamma^i \gamma^j R_{ijab}(\omega) \gamma^a \gamma^b \\
&= \frac{1}{\sqrt{g}} D_i^{(0)} \sqrt{g} g^{ij} D_j^{(0)} + \frac{1}{4} R
\end{aligned} \tag{111}$$

where $D^i D_i = g^{ij} D_i D_j = g^{ij} (D_i^{(0)} D_j - \Gamma_{ij}^k D_k)$ and $D_i^{(0)} = \partial_i + \frac{1}{4} \omega_{iab} \gamma^a \gamma^b$. We used the cyclic identity $R_{i[jab]} = 0$ to replace $\gamma^j \gamma^a \gamma^b$ by $e^{aj} \gamma^b - e^{bj} \gamma^a$, and then we used that the Ricci tensor R_{ia} is symmetric.

In the non-linear sigma model we choose the representation

$$\partial_i \leftrightarrow \frac{i}{\hbar} g^{1/4} p_i g^{-1/4}; \quad \gamma^i e_i^a = \gamma^a \leftrightarrow \sqrt{2} \psi^a \tag{112}$$

Then $\{\psi^a, \psi^b\} = \delta^{ab}$, and the Hamiltonian becomes

$$\hat{H} = \frac{1}{2} g^{-1/4} \left(p_i - \frac{\hbar i}{2} \omega_{iab} \psi^a \psi^b \right) g^{1/2} g^{ij} \left(p_j - \frac{\hbar i}{2} \omega_{jcd} \psi^c \psi^d \right) g^{-1/4} - \frac{\hbar^2}{8} R \tag{113}$$

The representation $p_i = g^{-1/4} \frac{\hbar}{i} \frac{\partial}{\partial x^i} g^{1/4}$ is fixed by the requirement that p_i be hermitian and that the inner product is given by $\langle x|y \rangle = \frac{1}{\sqrt{g(x)}} \delta(x-y)$, from which we find the completeness relations $\int d^n x \sqrt{g(x)} |x\rangle \langle x| = 1 = \int d^n p |p\rangle \langle p|$, and the inner product $\langle x|p \rangle = (2\pi\hbar)^{-n/2} \exp\left(\frac{i}{\hbar} p \cdot x\right) g^{-1/4}$. In the Fujikawa approach to anomalies, one often uses plane waves to evaluate traces in field theory. Since these plane waves are normalized to $\int \exp(\frac{i}{\hbar} p(x-y)) d^n p = (2\pi\hbar)^{n/2} \delta(x-y)$, the regulator is in these cases $\exp(-\tilde{\not{D}} \tilde{\not{D}} / M^2)$ with $\tilde{\not{D}} = g^{1/4} \not{D} g^{-1/4}$. In the non-linear sigma model we have inner products $\langle x|y \rangle = (1/\sqrt{g(x)}) \delta(x-y)$, and now the regulator corresponds to $\exp(-\not{D} \not{D} / M^2)$.

We recall that under Einstein transformations with anti-hermitian generator $E = \frac{-i}{2\hbar} \{p_j, F^j(x)\}$ for the orbital part, $x^i \rightarrow x^i + [x^i, E] = x^i + F^i(x) \equiv x'^i$, while the momenta transform as $p_i \rightarrow p'_i = \frac{1}{2} \left\{ \frac{\partial x^j}{\partial x'^i}, p_j \right\}$. Clearly ψ^a does not transform if we take x^i , p_i and ψ^a as independent variables. It follows that also $\pi_i = p_i - \frac{\hbar i}{2} \omega_{iab} \psi^a \psi^b$ transforms as $\pi'_i = \frac{1}{2} \left\{ \frac{\partial x^j}{\partial x'^i}, \pi_j \right\}$ (after adding a spin part to E which completes the transformation of ω_{iab} to that of a vector), and this shows, in the same way as in the bosonic case, that H is Einstein invariant. Since p_i is hermitian, H is clearly hermitian, too.

In a similar manner we may construct the orbital part of the Lorentz generator $L = \frac{1}{2\hbar} \lambda_{bc}(x) \psi^b \psi^c$ and show that $\delta_L p_j = \frac{1}{2i} (\partial_j \lambda_{bc}) \psi^b \psi^c$, $\delta_L \psi^a = \frac{1}{\hbar} \lambda^a_c \psi^c$ and so, after adding a spin part to L , $\delta_L (\omega_{iab} \psi^a \psi^b) = -(\partial_i \lambda_{ab}) \psi^a \psi^b$, from which also the Lorentz invariance of \hat{H} follows.

In the non-linear sigma model, $H = \frac{1}{2}QQ$, where $Q = \sqrt{2}\psi^a e_a^i g^{1/4} \pi_i g^{-1/4}$ is the supersymmetry generator. Since $[H, Q] = 0$, this Hamiltonian is supersymmetric. One can easily verify that Q is Einstein Lorentz invariant.

The chiral anomaly is given by

$$\mathcal{A}_n = \text{Tr} \gamma_5 e^{-\frac{\beta}{\hbar} \hat{H}}. \quad (114)$$

We shall compute this expression both in the case we double the number of Majorana fermions and in the case we combine the Majorana fermions into half as many Dirac fermions.

We start by doubling the number of fermions, in which case we evaluate the trace in an artificially extended Hilbert space. After we introduce the free fermions ψ_2^a ($a = 1 \dots n$), we have (for even n) $2^{n/2}$ states in the ψ_1 sector and $2^{n/2}$ states in the ψ_2 sector (combining the n ψ_1^a and n ψ_2^a in n pairs of creation and absorption operators). Hence, we must divide the trace over ψ_1 and ψ_2 by a factor $2^{n/2}$, since we really should only take the trace in the ψ_1 sector.

We shall now first express γ_5 into ψ^a . We define γ_5 by

$$\gamma_5 = (-i)^{n/2} \gamma_1 \gamma_2 \dots \gamma_n \quad n \text{ even} \quad (115)$$

So, in $n = 2$ one has $\gamma_5 = \tau_3$, and in $n = 4$ one has $\gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4$. In general $\gamma_5^2 = 1$ and γ_5 is hermitian. Identifying $\gamma^a = \sqrt{2} \hat{\psi}_1^a = (\hat{\psi}^a + \hat{\psi}_a^\dagger)$, $a = 1 \dots n$, we can evaluate the matrix element of γ_5 between fermionic coherent states and find

$$\langle \bar{\xi} | \gamma_5 | \eta \rangle = (-i)^{n/2} \langle \bar{\xi} | \eta \rangle \prod_{a=1}^n (\eta^a + \bar{\xi}^a) = (-i)^{n/2} \prod_{a=1}^n (\eta^a + \bar{\xi}^a) \quad (116)$$

The expression for the anomaly becomes (recall the factor of $2^{n/2}$)

$$\begin{aligned} \mathcal{A}_n = & 2^{-n/2} \int \prod_{i=1}^n dx_0^i \sqrt{g(x_0)} \int \prod_{a=1}^n d\bar{\eta}^a d\eta^a d\xi^a d\bar{\xi}^a \\ & e^{\bar{\xi}\xi} \langle \bar{\xi} | \gamma_5 | \eta \rangle e^{-\bar{\eta}\eta} \langle x_0, \bar{\eta} | e^{-\frac{\beta}{\hbar} \hat{H}} | x_0, \xi \rangle \end{aligned} \quad (117)$$

The last term in the above expression, the transition element, contains a factor $e^{\bar{\eta}\xi}$, and further contributions from loops (which depend only on the sum $(\xi^a + \bar{\eta}^a)$), but no classical action since the trace puts the initial and final points in x -space equal to each other. Now write

$$e^{\bar{\xi}\xi} e^{-\bar{\eta}\eta} \prod_{a=1}^n (\eta^a + \bar{\xi}^a) = e^{-\frac{1}{2}(\eta - \bar{\xi})(\xi - \bar{\eta})} \prod_{a=1}^n (\eta^a + \bar{\xi}^a) \quad (118)$$

and perform the integral over η and $\bar{\xi}$ (rewrite the measure $d\bar{\xi}^a d\eta^a$ in terms of the variables $\eta - \bar{\xi}$ and $\eta + \bar{\xi}$ as $2^n d(\bar{\xi}^a + \eta^a) d(\eta^a - \bar{\xi}^a)$). We find

$$\begin{aligned}\mathcal{A}_n &= \frac{(-i)^{n/2}}{(4\pi\beta\hbar)^{n/2}} \int \prod_{i=1}^n dx_0^i \sqrt{g(x_0)} \int \prod_{a=1}^n d\bar{\eta}^a d\xi^a e^{\bar{\eta}\xi} \prod_{a=1}^n (\xi^a - \bar{\eta}^a) e^{-\frac{1}{\hbar} S_{\text{loops}}(x_0, \bar{\eta} + \xi)} \\ &= \frac{(-i)^{n/2}}{(2\pi\beta\hbar)^{n/2}} \int \prod_{i=1}^n dx_0^i \sqrt{g(x_0)} \int \prod_{a=1}^n d\psi_{\text{bg}}^a e^{-\frac{1}{\hbar} S_{\text{loops}}(x_0, \psi_{\text{bg}}^a)}\end{aligned}\quad (119)$$

We were able to integrate out the ψ_2^a , canceling the factor $2^{-n/2}$, because S_{loops} only depends on the combination $\psi_{\text{bg}}^a = \frac{1}{\sqrt{2}}(\xi^a + \bar{\eta}^a)$, which is the background value of ψ_1^a . The factor $(2\pi\hbar\beta)^{n/2}$ normalizes the leading (classical) singularity in $\langle x | \exp(-\frac{\beta}{\hbar} H) | x \rangle$ to a Dirac delta function. The loop contributions $S_{\text{loops}}(x_0, \psi_{\text{bg}}^a)$ are defined by

$$e^{\bar{\eta}\xi - \frac{1}{\hbar} S_{\text{loops}}(x_0, \psi_{\text{bg}}^a)} = \left\langle e^{-\frac{1}{\hbar} S_{\text{int}}} \right\rangle \quad (120)$$

with the propagators given in (80). S_{int} is the interaction part of the action S

$$\begin{aligned}S &= -\hbar\delta_{ab}\bar{\psi}^a(0)\psi^b(0) + \int_{-\beta}^0 dt \left[\frac{1}{2}g_{ij}(x) \left(\dot{x}^i \dot{x}^j + \frac{1}{\beta^2} b^i c^j + \frac{1}{\beta^2} a^i a^j \right) \right. \\ &\quad \left. + \hbar\delta_{ab}\bar{\psi}^a \dot{\psi}^b + \frac{\hbar}{2} \dot{x}^i \omega_{iab} \psi_1^a \psi_1^b + (\omega\omega \text{ and } \Gamma\Gamma \text{ terms}) \right]\end{aligned}\quad (121)$$

with the fields subject to the boundary conditions $x^i(0) = x^i(-1) = x_0^i$ and $\bar{\psi}^a(0) = \bar{\eta}^a$, $\psi^a(-1) = \xi^a$. We now rescale $t = \beta\tau$, which leads in the bosonic sector to

$$\frac{1}{\hbar} S_{\text{bos}} = \frac{1}{\beta\hbar} \int_{-1}^0 d\tau \frac{1}{2} g_{ij}(x) \left(\dot{x}^i \dot{x}^j + b^i c^j + a^i a^j \right) \quad (122)$$

To obtain also a factor $\frac{1}{\beta}$ in front of the fermionic terms we rescale ψ and $\bar{\psi}$ suitably

$$\bar{\psi}^a = \frac{1}{\sqrt{\beta}} \bar{\psi}'^a, \quad \psi^a = \frac{1}{\sqrt{\beta}} \psi'^a \quad (123)$$

Then

$$\frac{1}{\hbar} S_{\text{fer}} = \frac{1}{\beta} \int_{-1}^0 d\tau \left(\delta_{ab} \bar{\psi}'^a \dot{\psi}'^b + \frac{1}{2} \dot{x}^i \omega_{iab} \psi'^a \psi'^b \right) - \frac{1}{\beta} \delta_{ab} \bar{\psi}'^a(0) \psi'^b(0) \quad (124)$$

The same rescaling is applied to the corresponding background values

$$\bar{\eta}^a = \frac{1}{\sqrt{\beta}} \bar{\eta}'^a, \quad \chi^a = \frac{1}{\sqrt{\beta}} \chi'^a \quad (125)$$

With this rescaling of the fermionic background fields, the β dependence in the measure in (119) is also canceled. Dropping the primes, we arrive at

$$\mathcal{A}_n = \frac{(-i)^{n/2}}{(2\pi\hbar)^{n/2}} \int \prod_{i=1}^n dx_0^i \sqrt{g(x_0)} \int \prod_{a=1}^n d\psi_{\text{bg}}^a e^{-\frac{1}{\hbar} S_{\text{loops}}(x_0, \psi_{\text{bg}}^a)} \quad (126)$$

with the action the sum of (122) and (124). The $\Gamma\Gamma$ and $\omega\omega$ term are of order β (due to the rescaling $t = \beta\tau$) and play no further role in the evaluation of chiral anomalies.

Expanding

$$x^i = x_0^i + q^i, \quad \psi^a = \chi^a + \psi_{\text{qu}}^a, \quad \bar{\psi}^a = \bar{\eta}^a + \bar{\psi}_{\text{qu}}^a \quad (127)$$

we see that

- (i) all vertices are proportional to $\frac{1}{\beta}$ and all propagators are proportional to β .
- (ii) hence only one-loop graphs contribute.
- (iii) In a frame where $\omega_{iab}(x_0) = 0$, there are no terms linear in the quantum fields in S , so no tadpoles. Therefore, only vertices with exactly two quantum fields are relevant for the one-loop graphs in that case.
- (iv) Consequently, by expanding $g_{ij}(x)$ and $\omega_{iab}(x)$ around x_0 , and after substituting (127) in S , we find that in the frame $\omega_{iab}(x_0) = 0$ the only vertices are

$$\frac{1}{\hbar} S_{\text{int}} = \frac{1}{2\beta} (\partial_j \omega_{iab}(x_0)) \int_{-1}^0 d\tau q^j \dot{q}^i \psi_{\text{bg}}^a \psi_{\text{bg}}^b \quad (128)$$

which can be rewritten as

$$\frac{1}{\hbar} S_{\text{int}} = \frac{1}{4\beta} R_{ijab}(\omega(x_0)) \psi_{\text{bg}}^a \psi_{\text{bg}}^b \int_{-1}^0 d\tau q^i \dot{q}^j \quad (129)$$

Thus the one-loop contributions are due to a q loop, with at each vertex two ψ_{bg}^a sticking out.

The one-loop result can now easily be evaluated by noting that it is proportional to the one-loop determinant

$$\mathcal{A}_n = \frac{(-i)^{n/2}}{(2\pi\hbar)^{n/2}} \int d^n x_0 \sqrt{g(x_0)} \prod_{a=1}^n d\psi_{\text{bg}}^a \left[\frac{\det \left(-\frac{d^2}{d\tau^2} g_{ij}(x_0) + \frac{1}{2} R_{ijab} \psi_{\text{bg}}^a \psi_{\text{bg}}^b \frac{d}{d\tau} \right)}{\det \left(-\frac{d^2}{d\tau^2} g_{ij}(x_0) \right)} \right]^{-1/2} \quad (130)$$

The denominator in this expression is the ghost contribution to \mathcal{A}_n . The factor $i^{n/2}$ can be removed by a rescaling of ψ_{bg}^a , in which case the only thing that changes is that R_{ijab} is replaced by iR_{ijab} . With this rescaling the operator in the numerator of (130) is hermitian and it becomes manifest that \mathcal{A}_n is real. From here on, one can follow Alvarez-Gaumé and Witten [2]. First one skew diagonalizes R_{ij} , then one computes the eigenvalues of the operator in the determinant. The product of the corresponding

eigenvalues then immediately yields the well-known result for the chiral anomaly as the \hat{A} -genus of the manifold. An alternative derivation can be found in appendix A.4, where the one-loop diagrams contributing to the one-loop determinant (130) are explicitly evaluated.

We will now evaluate the trace in (114) in case we do not double the number of fermions, but instead combine the Majorana fermions into half as many Dirac fermions. In this case the expression corresponding to γ_5 becomes

$$\gamma_5 = (-i)^{n/2} \prod_{A=1}^{n/2} (\Psi^A + \bar{\Psi}^A) \frac{1}{i} (\Psi^A - \bar{\Psi}^A) = \prod_{A=1}^{n/2} (1 - 2\bar{\Psi}^A \Psi^A) \quad (131)$$

where the Ψ^A and $\bar{\Psi}^A$ are defined as in (54) and we identify again $\gamma^a = \sqrt{2}\psi^a$. Since $P_A = \bar{\Psi}^A \Psi^A$ for fixed A is a projection operator ($P_A^2 = P_A$) we can rewrite this as

$$\gamma_5 = \prod_{A=1}^{n/2} e^{-i\pi \bar{\Psi}^A \Psi^A} = \exp\left(-i\pi \sum_{A=1}^{n/2} \bar{\Psi}^A \Psi^A\right) \quad (132)$$

since expansion gives $\prod_A (1 + (e^{-i\pi} - 1)\bar{\Psi}^A \Psi^A)$.

The matrix element of γ_5 between coherent states simplifies even further

$$\begin{aligned} \langle \bar{\xi} | \gamma_5 | \eta \rangle &= e^{\bar{\xi} \eta} \prod_{A=1}^{n/2} (1 - 2\bar{\xi}^A \eta^A) = \prod_{A=1}^{n/2} (1 - 2\bar{\xi}^A \eta^A) (1 + \bar{\xi}^A \eta^A) \\ &= \prod_{A=1}^{n/2} (1 - \bar{\xi}^A \eta^A) = \exp\left(-\sum_{A=1}^{n/2} \bar{\xi}^A \eta^A\right) \end{aligned} \quad (133)$$

To evaluate the anomaly, we put together the expression for the Jacobian and the transition element

$$\begin{aligned} \mathcal{A}_n &= \int \prod_{i=1}^n dx_0^i \sqrt{g(x_0)} \int \prod_{A=1}^{n/2} d\bar{\eta}^A d\eta^A d\xi^A d\bar{\xi}^A \\ &\quad e^{\bar{\xi} \xi} \langle \bar{\xi} | \gamma_5 | \eta \rangle e^{-\bar{\eta} \eta} \langle x_0, \bar{\eta} | e^{-\frac{\beta}{\hbar} \hat{H}} | x_0, \xi \rangle \end{aligned} \quad (134)$$

As before, we extract a factor $\exp \bar{\eta} \xi$ from the transition element, and write

$$\begin{aligned} \mathcal{A}_n &= \frac{1}{(2\pi\beta\hbar)^{n/2}} \int \prod_{i=1}^n dx_0^i \sqrt{g(x_0)} \int \prod_{A=1}^{n/2} d\bar{\eta}^A d\eta^A d\xi^A d\bar{\xi}^A \\ &\quad e^{\bar{\xi} \xi - \bar{\xi} \eta - \bar{\eta} \eta + \bar{\eta} \xi} \exp\left(-\frac{1}{\hbar} S_{\text{loops}}(x_0, \bar{\eta}^A, \xi^A)\right) \end{aligned} \quad (135)$$

We can now trivially do the integral over η^A and $\bar{\xi}^A$, after which the above expression equals (119), with the identifications

$$\begin{aligned} \psi_{\text{bg}}^a &= \frac{1}{\sqrt{2}} (\xi^{(a+1)/2} + \bar{\eta}^{(a+1)/2}) \quad a \text{ odd} \\ \psi_{\text{bg}}^a &= -\frac{i}{\sqrt{2}} (\xi^{a/2} - \bar{\eta}^{a/2}) \quad a \text{ even} \end{aligned} \quad (136)$$

Hence, also in this case, we obtain the same expression for the chiral anomaly.

3.2 Chiral anomalies in Yang-Mills couplings

We consider complex spin-1/2 fermions coupled to external Yang-Mills gauge fields corresponding to a group G , with the fermions transforming under a representation R . We will only consider flat space here; by combining the techniques in this and the previous section, one can easily obtain the combined gravitational and Yang-Mills anomalies. We leave this as an exercise to the reader. New in this section is the treatment of the Yang-Mills ghosts by path integrals. In [2], they were kept as operators.

The Jacobian is still γ_5 , but the regulator is now proportional to

$$\not{D} \not{D} = D^\mu D_\mu + \frac{1}{2} \gamma^\mu \gamma^\nu [D_\mu, D_\nu], \quad (137)$$

where $D_\mu = \partial_\mu - g A_\mu^\alpha T_\alpha$, $[T_\alpha, T_\beta] = f_{\alpha\beta}^\gamma T_\gamma$ and $[D_\mu, D_\nu] = -g F_{\mu\nu}^\alpha T_\alpha$. Hence, compared to the bosonic case, there is now an extra term proportional to the Yang-Mills curvature. We represent the matrices $(T_\alpha)_N^M$, $M, N = 1 \dots \dim R$, where $\dim R$ is the dimension of the representation R , by operators

$$\hat{T}_\alpha = c_M^* (T_\alpha)_N^M c^N, \quad (138)$$

and require the anticommuting relations

$$\{c^N, c_M^*\} = \delta_M^N. \quad (139)$$

Then $[\hat{T}_\alpha, \hat{T}_\beta] = f_{\alpha\beta}^\gamma \hat{T}_\gamma$. The Dirac matrices γ^a are again represented by $\sqrt{2}\psi^a$ with

$$\{\psi^a, \psi^b\} = \delta^{ab} \quad (a, b = 1, \dots, n) \quad (140)$$

We will only represent the Majorana fermions ψ^a by half as many Dirac fermions here; if one doubles the number of fermions one obtains the same answer. This can easily be demonstrated in a similar fashion as we did in the previous section. With half as many Dirac fermions we had (132)

$$\gamma_5 \rightarrow e^{-i\pi\bar{\psi}_A \psi^A}. \quad (141)$$

The anomaly is then represented in the non-linear sigma model by

$$\mathcal{A}_n = \text{Tr}' e^{-i\pi\bar{\psi}_A \psi^A} e^{-\beta H/\hbar} \quad (142)$$

$$\hat{H} = \frac{1}{2} (p_j + \hbar i A_j^\alpha c^* T_\alpha c) (p_k + \hbar i A_k^\beta c^* T_\beta c) \delta^{jk} + \frac{\hbar^2}{2} \psi^a \psi^b F_{ab}^\alpha c^* T_\alpha c \quad (143)$$

We have rescaled $A_j^\alpha(x)$ and $F_{\mu\nu}^\alpha$ such that the coupling constants have been absorbed.

The prime on the trace, Tr' , indicates that the trace is not over all states in the Fock space (which are $|0\rangle, c_{M_1}^*|0\rangle, c_{M_1}^*c_{M_2}^*|0\rangle, \dots, c_{M_1}^* \cdots c_{M_{\text{dimR}}}^*|0\rangle$, but rather only over the one-particle states $c_M^*|0\rangle$. Only on these states does $c^*T_\alpha c$ act like the matrix T_α . To still write the trace as an unconstrained trace, we introduce the one-particle projection operator P . We claim that

$$P =: x e^{-x} :, \quad x = c_M^* c^M. \quad (144)$$

Indeed, from the definition of x it follows that

$$: x^n := (x - n + 1) : x^{n-1} := \dots = n! \binom{x}{n}. \quad (145)$$

Then we use a representation of the Kronecker delta $\delta_{x,1}$,

$$\delta_{x,1} = \lim_{p \rightarrow 1} x p (1-p)^{x-1} = \lim_{p \rightarrow 1} p \frac{\partial}{\partial p} [-(1-p)^x]. \quad (146)$$

Expanding $(1-p)^x$ in a power series and performing the limit $p \rightarrow 1$ we arrive at

$$\sum_{n \geq 0} n (-1)^{n-1} \binom{x}{n} = \sum_{n > 0} [n! \binom{x}{n}] \frac{(-1)^{n-1}}{(n-1)!} \quad (147)$$

Using (145) we get the desired result, namely

$$P =: x e^{-x} := \delta_{x,1}. \quad (148)$$

The anomaly is thus given by

$$\mathcal{A}_n = \text{Tr} \left(e^{-i\pi \bar{\psi}_A \psi^A} : c_M^* c^M e^{-c_N^* c^N} : e^{-\beta H/\hbar} \right). \quad (149)$$

Using complete sets of coherent states, one finds the corresponding path integral representation

$$\mathcal{A}_n = \text{tr}_x \text{tr}_f \text{tr}_{gh} \langle x_0, \bar{\chi}_{gh}, \bar{\chi}_f | e^{-i\pi \bar{\psi} \psi} P I_{gh} I_f e^{-\beta H/\hbar} | \chi_f, \chi_{gh}, x_0 \rangle, \quad (150)$$

where

$$I_{gh} = \int d\bar{\eta}_{gh} d\eta_{gh} |\eta_{gh}\rangle e^{-\bar{\eta}_{gh} \eta_{gh}} \langle \bar{\eta}_{gh}|, \quad (151)$$

$$I_f = \int d\bar{\eta}_f d\eta_f |\eta_f\rangle e^{-\bar{\eta}_f \eta_f} \langle \bar{\eta}_f|, \quad (152)$$

$$\text{tr}_x = \int dx_0 \quad (153)$$

$$\text{tr}_{gh} = \int d\chi_{gh} d\bar{\chi}_{gh} e^{\bar{\chi}_{gh} \chi_{gh}}, \quad (154)$$

$$\text{tr}_f = \int d\chi_f d\bar{\chi}_f e^{\bar{\chi}_f \chi_f}. \quad (155)$$

We have used that in fermionic spaces the trace of an operator is given by (32) and further inserted two unit operators I for which we used the decomposition in terms of coherent states.

The trace involving $e^{-i\pi\bar{\psi}\psi}P$ factorizes. The ghost dependent part of \mathcal{A}_n reads

$$\int d\chi_{gh} d\bar{\chi}_{gh} d\bar{\eta}_{gh} d\eta_{gh} e^{\bar{\chi}_{gh}\chi_{gh}} e^{-\bar{\eta}_{gh}\eta_{gh}} \langle \bar{\chi}_{gh} | : xe^{-x} : | \eta_{gh} \rangle \langle \bar{\eta}_{gh} | e^{-\beta H} | \chi_{gh} \rangle. \quad (156)$$

Since P is a one-particle projection operator its matrix element simply yields

$$\langle \bar{\chi}_{gh} | : xe^{-x} : | \eta_{gh} \rangle = \bar{\chi}_{gh}\eta_{gh}, \quad (157)$$

The integration over $\bar{\chi}_{gh}$ then yields

$$\begin{aligned} \int d\bar{\chi}_{gh}^{\text{dimR}} \dots d\bar{\chi}_{gh}^1 \left(\sum_M \bar{\chi}_{gh,M} \eta_{gh}^M \right) e^{\bar{\chi}_{gh}\chi_{gh}} &= \sum_M \left(\left(\prod_{N>M} \chi_{gh}^N \right) \eta_{gh}^M \left(\prod_{N<M} \chi_{gh}^N \right) \right) \\ &= \sum_M \chi_{gh}^{\text{dimR}} \dots \eta_{gh}^M \dots \chi_{gh}^1, \end{aligned} \quad (158)$$

namely, a product of all χ_{gh}^N except that the M -th factor is replaced by η_{gh}^M . The integration over η_{gh} then yields

$$\int d\eta_{gh} e^{-\bar{\eta}_{gh}\eta_{gh}} \sum_M \left(\left(\prod_{N>M} \chi_{gh}^N \right) \eta_{gh}^M \left(\prod_{N<M} \chi_{gh}^N \right) \right) = \sum_M \prod_{N \neq M} (\bar{\eta}_{gh,N} \chi_{gh}^N). \quad (159)$$

Clearly, this operator projects an arbitrary function of $\bar{\eta}_{gh}$ and χ_{gh} onto the terms with precisely one $\bar{\eta}_{gh}$ and one χ_{gh} .

In the fermionic sector, the matrix element of γ_5 is again (133)

$$\langle \bar{\chi}_f | e^{-i\pi\bar{\psi}_A\psi^A} | \eta_f \rangle = e^{-\bar{\chi}_f\eta_f}. \quad (160)$$

This leads to the integral

$$\int d\bar{\chi}_f d\eta_f e^{\bar{\chi}_f\eta_f} e^{-\bar{\eta}_f\eta_f} e^{-\bar{\chi}_f\eta_f} = e^{-\bar{\eta}_f\chi_f}. \quad (161)$$

Inserting these subresults from the ghost and fermionic sectors into the anomaly equation we arrive at

$$\begin{aligned} \mathcal{A}_n &= \int \prod_{i,M,A} [dx_0^i d\chi_{gh}^M d\bar{\eta}_{gh,M} d\bar{\eta}_{f,A} d\chi_f^A] \\ &\quad \times \left[\sum_M \left(\prod_{N \neq M} \bar{\eta}_{gh,N} \chi_{gh}^N \right) \right] e^{-\bar{\eta}_f\chi_f} \langle x_0, \bar{\eta}_{gh}, \bar{\eta}_f | e^{-\beta H/\hbar} | \chi_f, \chi_{gh}, x_0 \rangle. \end{aligned} \quad (162)$$

The transition element contains a factor $e^{\bar{\eta}_f\chi_f}$ and a factor $e^{\bar{\eta}_{gh}\chi_{gh}}$, in addition to contributions from loops. The former cancels against (161), so that we obtain

$$\begin{aligned} \mathcal{A}_n &= \int \prod_i \left[\frac{dx_0^i}{\sqrt{2\pi\hbar\beta}} \right] \int \prod_{M,A} [d\chi_{gh}^M d\bar{\eta}_{gh,M} d\bar{\eta}_{f,A} d\chi_f^A] \left[\sum_M \left(\prod_{N \neq M} \bar{\eta}_{gh,N} \chi_{gh}^N \right) \right] e^{\bar{\eta}_{gh}\chi_{gh}} \\ &\quad \times \exp \left(-\frac{1}{\hbar} S_{\text{loops}}(x_0, \bar{\eta}_{gh}, \bar{\eta}_f, \chi_{gh}, \chi_f) \right), \end{aligned} \quad (163)$$

where

$$e^{\bar{\eta}_f \chi_f + \bar{\eta}_{gh} \chi_{gh} - \frac{1}{\hbar} S_{\text{loops}}(x_0, \bar{\eta}_{gh}, \bar{\eta}_f, \chi_{gh}, \chi_f)} = \left\langle e^{-\frac{1}{\hbar} S_{\text{int}}} \right\rangle \quad (164)$$

Here, S_{int} is the interaction part of the action

$$\begin{aligned} \frac{1}{\hbar} S &= \frac{1}{\hbar} \int_{-\beta}^0 dt \left(\frac{1}{2} \left(\frac{dx^i}{dt} \right)^2 + \hbar \bar{\psi}_A \frac{d\psi^A}{dt} + \hbar c_M^* \frac{dc^M}{dt} \right) - \bar{\psi}_A(0) \psi^A(0) - c_M^*(0) c^M(0) \\ &\quad - \int_{-\beta}^0 dt (\dot{x}^j A_j^\alpha c_M^*(T_\alpha)^M{}_N c^N - \frac{\hbar}{2} \psi^a \psi^b F_{ab}^\alpha c_M^*(T_\alpha)^M{}_N c^N), \end{aligned} \quad (165)$$

The couplings $\dot{x} A c^* T c$ result from integrating out the momenta p , and combine with the ghost kinetic term to the covariant derivative $D_t c^M = \dot{c}^M - \dot{x}^j A_j^\alpha(x) T_\alpha^M{}_N c^N$. The fields satisfy the boundary conditions $x(0) = x(-\beta) = x_0$, $\bar{\psi}_f(0) = \bar{\eta}_f$, $c^*(0) = \bar{\eta}_{gh}$, $\psi_f(-\beta) = \chi_f$, and $c(-\beta) = \chi_{gh}$. Decomposing all fields into background parts and quantum fluctuations as in (127), we note that the object S_{loops} now contains tree graphs because the four-fermion couplings $\psi \chi c^* c$ contain terms linear in quantum fields and hence lead to tree graphs. These tree graphs do, of course, contribute to the classical action, as we discussed in section 2.

In order to study the loop expansion in detail, we rescale $t = \beta\tau$, and $\psi \rightarrow \psi'(\beta\hbar)^{-1/2}$, $\bar{\psi} \rightarrow \bar{\psi}'(\beta\hbar)^{-1/2}$, $\chi_f \rightarrow \chi_f'(\beta\hbar)^{-1/2}$, $\bar{\eta}_f \rightarrow \bar{\eta}_f'(\beta\hbar)^{-1/2}$. After the rescaling of $d\chi_f$ and $d\bar{\eta}_f$ the measure becomes completely $\beta\hbar$ -independent, and

$$\begin{aligned} \frac{1}{\hbar} S &= \frac{1}{\beta\hbar} \int_{-1}^0 d\tau \left(\frac{1}{2} \left(\frac{dx^i}{d\tau} \right)^2 + \bar{\psi}_A \frac{d\psi^A}{d\tau} \right) - \frac{1}{\beta\hbar} \bar{\psi}_A(0) \psi^A(0) - c_M^*(0) c^M(0) \\ &\quad + \int_{-1}^0 d\tau (c_M^* \dot{c}^M - \dot{x}^j A_j^\alpha c_M^*(T_\alpha)^M{}_N c^N + \frac{1}{2} \psi^a \psi^b F_{ab}^\alpha c_M^*(T_\alpha)^M{}_N c^N) \end{aligned} \quad (166)$$

Since the x and ψ propagators are proportional to $\beta\hbar$ and all vertices and $c^* c$ propagators are $\beta\hbar$ -independent, we need not consider the vertices containing the q^i , $\bar{\psi}_{\text{qu},A}$ or ψ_{qu}^A of (127). Hence, we can restrict our attention to the vertices $F\psi\psi c^* T c$, with the ψ 's replaced by their background value. There are no classical contributions to S_{loops} since we evaluate the classical action from y to z with $y = z = x_0$. So altogether we are left with

$$\begin{aligned} \mathcal{A}_n &= \int \prod_{i,M,A} \left[\frac{dx_0^i}{\sqrt{2\pi\hbar}} d\chi_{gh}^M d\bar{\eta}_{gh,M} d\bar{\eta}_{f,A} d\chi_f^A \right] \left[\sum_M \left(\prod_{N \neq M} \bar{\eta}_{gh,N} \chi_{gh}^N \right) \right] e^{\bar{\eta}_{gh} \chi_{gh}} \\ &\quad \left\langle \exp \left[- \int_{-1}^0 d\tau (c_M^* \dot{c}^M + \frac{1}{2} \psi_{\text{bg}}^a \psi_{\text{bg}}^b F_{ab}^\alpha c^* T_\alpha c) \right] \right\rangle_{\text{loops}} \end{aligned} \quad (167)$$

where ψ_{bg}^a are the classical values of ψ^a exactly as in (136).

The propagator $\langle c^M(\sigma) c_N^*(\tau) \rangle$ is equal to $\delta_N^M \theta(\sigma - \tau)$, hence no closed c -loops can contribute (in a closed c -loop one always moves somewhere backwards in time).

Only terms with precisely one $\bar{\eta}_{gh}$ and one χ_{gh} can contribute due to the projection operator in the measure. These terms are just tree graphs, with $c^* = \bar{\eta}_{gh}$ at one end and $c = \chi_{gh}$ at the other end. If we consider a diagram with k vertices, then it will contain an integration

$$\int_{-1}^0 d\sigma_1 \cdots d\sigma_k \theta(\sigma_1 - \sigma_2) \cdots \theta(\sigma_{k-1} - \sigma_k) = \frac{1}{k!}. \quad (168)$$

The combinatorical factor $1/k!$ from the expansion of $e^{-S/\hbar}$ cancels because there are exactly $k!$ different ways to build a connected tree graph from the vertices. Putting all this together yields

$$-\frac{1}{\hbar} S_{\text{loops}} = \bar{\eta}_{gh,M} \left(e^{-\frac{1}{2} F_{ab}^\alpha \psi_{\text{bg}}^a \psi_{\text{bg}}^b T_\alpha} - 1 \right)^M {}_N \chi_{gh}^N \quad (169)$$

The background term $\exp(\bar{\eta}_{gh,M} \chi_{gh}^M)$ in (167) cancels the -1 in (169). The integration over $\bar{\eta}_{gh}$ and χ_{gh} in (167) with S_{loops} replaced by (169) can now easily be done, since it picks out the piece with only one $\bar{\eta}_{gh}$ and χ_{gh} from S_{loops} . Transforming in addition variables from $\bar{\eta}_f$ and χ_f to ψ_{bg}^a leaves us then finally with the following result

$$\begin{aligned} \mathcal{A}_n &= \left(\frac{-i}{2\pi} \right)^{n/2} \int dx_0^i d\psi_{\text{bg}}^1 \cdots d\psi_{\text{bg}}^n \text{tr} \left(e^{-\frac{1}{2} F_{ab}^\alpha \psi_{\text{bg}}^a \psi_{\text{bg}}^b T_\alpha} \right) \\ &= \left(\frac{i}{4\pi} \right)^{n/2} \frac{1}{\left(\frac{n}{2} \right)!} \int dx_0^i \epsilon^{a_n \cdots a_1} \text{tr} (F_{a_1 a_2} \cdots F_{a_{n-1} a_n}) \end{aligned} \quad (170)$$

where the trace is over the Yang-Mills indices of T_α in $F_{ab} = F_{ab}^\alpha T_\alpha$. This is the correct anomaly.

3.3 Trace anomalies

We can now easily compute the trace anomalies for a spin-0 and spin- $\frac{1}{2}$ field in an n -dimensional quantum field theory. In this case, the Jacobian is equal to one, so we just need to evaluate the trace of the appropriate transition element, and no additional operators need to be inserted. The anomaly is then equal to the β -independent part of the trace. New in this section is the treatment of fermions by path integrals; in [9] they were kept as operators.

For a real scalar field in $n = 2$ dimensions, we can directly take the trace in (11). Singling out the β -independent part yields

$$A_2^{\text{trace}} = -\frac{\hbar}{24\pi} R \quad (171)$$

In order to obtain the trace anomaly for a scalar field in higher dimensions, we need to compute the terms of higher order in β in the transition element. Since the transition element contains a factor $(2\pi\beta\hbar)^{-n/2}$, one needs $\frac{n}{2} + 1$ loops in n -dimensional space.

We will now evaluate the trace anomaly for a spin- $\frac{1}{2}$ field in n dimensions. In this case we have to compute the trace of the N=1 supersymmetric transition element, and project out the β -independent part. As claimed before, both approaches to obtain fermionic creation and annihilation operators will lead to the same result.

Recall that when we double the number of fermions we should normalize the trace by dividing by an extra factor $2^{n/2}$. The trace anomaly for a spin- $\frac{1}{2}$ field in n dimensions is therefore given by (cf. [9], equation (2.9))

$$\mathcal{A}_n^{spin-\frac{1}{2}} = -\frac{1}{2^{n/2}} \lim_{\beta \rightarrow 0} \int d\chi^a d\bar{\eta}^a e^{\bar{\eta}\chi} \langle x_0, \bar{\eta} | \exp\left(-\frac{\beta}{\hbar} \hat{H}\right) | x_0, \chi \rangle \quad ; \quad a = 1 \dots n \quad (172)$$

with the transition element given in (81). For $n = 2$ the trace anomaly becomes $\frac{\hbar}{24\pi} R$, which is indeed the result for a Dirac fermion; for the anomaly of a Majorana fermion we have to divide this expression by two.

When we instead combine the Majorana fermions into half as many Dirac fermions, we should take the trace of the transition element given in (96). In this case of course no extra normalization factor is needed, and we find directly

$$\mathcal{A}_n^{spin-\frac{1}{2}} = -\hbar \lim_{\beta \rightarrow 0} \int d\chi^A d\bar{\eta}^A e^{\bar{\eta}\chi} \langle x_0, \bar{\eta} | \exp\left(-\frac{\beta}{\hbar} \hat{H}\right) | x_0, \chi \rangle \quad ; \quad A = 1 \dots n/2 \quad (173)$$

now with the transition element given in (96). This gives again the same result for the anomaly as above.

4 Conclusions

In this article we have given a complete, explicit derivation of quantum mechanical path integrals for bosons and fermions, both for Dirac fermions and Majorana fermions. Our main result is that the factors $\delta(\sigma - \tau)$ in the Feynman rules for configuration space path integrals should be interpreted as Kronecker delta functions, even in the continuum case, and should not be regulated by mode regularization.

We *define* our path integrals in curved space by starting from the Hamiltonian (operator) formalism. After inserting complete sets of states (coherent states for fermions), and Weyl-ordering the Hamiltonian (leading to order \hbar and \hbar^2 terms in the path integral action), we obtained the discretized propagators and vertices *in closed form*. With these ingredients one can construct a loop expansion of the path integral in terms of Feynman integrals. Some of the bosonic propagators contain $\delta(\sigma - \tau)$

singularities, but adding ‘Lee-Yang ghosts’, terms with two or more $\delta(\sigma - \tau)$ cancel. Terms with one $\delta(\sigma - \tau)$ should then be evaluated as indicated above.

We paid particular attention to Majorana fermions. Of course, starting with an arbitrary initial state $|A\rangle$ and acting on it with products of ψ ’s, the states so obtained will span a Hilbert space on which the ψ ’s can be represented as matrices. One can then define matrix-valued Hamiltonians as in [9]. We found it much simpler to define creation and annihilation operators and then to use the standard formulation of coherent states [21]. The particular way of defining creation and annihilation operators is, of course, arbitrary, and so is therefore the choice of vacuum, but in problems involving a trace over the Hilbert space, this arbitrariness should cancel. We achieved the construction of creation and annihilation operators in two ways: either by combining the Majorana spinors pairwise into creation and annihilation operators (‘halving’), or by adding another set of Majorana spinors (‘doubling’, this works also for odd-dimensional spaces). Of course, the Hilbert spaces, vacua etc. are different in both cases, and indeed we found different expressions for the transition element, but the anomalies came out the same. This confirms our claim that in traces over the Hilbert spaces, differences created by choosing different vacua should cancel. We verified our formalism by computing the transition elements for a bosonic and several fermionic transition elements through order β , and comparing the results with those from (unambiguous) operator calculations. We provided further evidence by doing a three-loop calculation in appendix A.1.

We applied our general formalism to trace anomalies and to chiral anomalies for spin-1/2 fields. The chiral anomalies were already studied by Alvarez-Gaumé and Witten and the trace anomalies in [9], but we have treated the Yang-Mills ghosts and the Majorana fermions on equal footing with the bosons and obtained a uniform path integral treatment. Moreover, our derivation makes a detailed and complete treatment for any anomaly possible, including all normalizations.

On the more technical side, we have seen that the Hamiltonian \hat{H} in the operator formalism and the Hamiltonian function H in the path integral are related by the formula $\hat{H} = (H)_W$, where $(H)_W$ contains in general terms of order \hbar and \hbar^2 . If \hat{H} is Einstein invariant, the corresponding action in the path integral is not Einstein invariant (although the transition element is); rather there are noncovariant $\Gamma\Gamma$ terms. Conversely, if the action is the naive action, \hat{H} will contain \hbar and \hbar^2 terms. In particular, the supersymmetric Hamiltonians (whose ambiguity was fixed by requiring hermiticity and Einstein invariance) do not lead to the usual classically supersymmetric action in the path integral. Rather, there are extra terms proportional to \hbar^2 . For chiral anomalies, though, these extra terms in the action do not contribute, which

explains why the results of Alvarez-Gaumé and Witten [2] are correct (that they are correct can be checked by doing loop calculations in the corresponding quantum field theory, see [2] and [38]). For trace anomalies, the extra terms do matter. Here higher loop calculations are needed and we stressed that the noncovariant vertices (of the form $\Gamma\Gamma$ and $\omega\omega$) as well as our new Feynman rules *must* be taken into account, even when one uses normal co-ordinates, to obtain the correct results.

This concludes our analysis of quantum mechanical path integrals. One might wonder whether the subtleties we have found in one dimension have a counterpart in higher dimensions. For local field theories this seems unlikely, because we expect that possible extra terms in the path integral will be proportional to $\delta^{n-1}(\vec{x})$, and hence would vanish in dimensional regularization. However, in non-local field theories, such as Yang-Mills theories in the Coulomb gauge [13, 15], there might be effects. This is under study.

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A Appendices

A.1 A 3-loop computation

We check that there are no counterterms beyond the two-loop counterterm in (13) by performing a 3-loop calculation in a model which describes a free particle, but which has been cast into the form of a non-linear sigma model by a nontrivial co-ordinate transformation. This extends the two-loop phase space calculation of [14]. We shall then redo the calculation in configuration space and show that we obtain the same result (Matthews' theorem).

We consider a free massive point particle on the interval $-\infty < t < \infty$ with $L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2$ and substitute $q = Q + \frac{1}{3}Q^3$. Then the action becomes

$$L(Q, \dot{Q}) = \frac{1}{2}\dot{Q}^2(1 + Q^2)^2 - \frac{1}{2}Q^2(1 + \frac{1}{3}Q^2)^2 \quad (174)$$

and the Hamiltonian is $H(Q, P) = \frac{1}{2}P^2(1 + Q^2)^{-2} + \frac{1}{2}Q^2(1 + \frac{1}{3}Q^2)^2$. The first-order action in phase space reads

$$L(Q, P) = (P\dot{Q} - \frac{1}{2}P^2 - \frac{1}{2}Q^2) + \mathcal{L}_{int}(Q, P)$$

$$\begin{aligned}
L_{int}(Q, P) &= \frac{1}{2}P^2(2Q^2 + Q^4)(1 + Q^2)^{-2} - \frac{1}{3}Q^4 - \frac{1}{18}Q^6 \\
&= P^2Q^2 - \frac{3}{2}P^2Q^4 - \frac{1}{3}Q^4 - \frac{1}{18}Q^6 + \dots
\end{aligned} \tag{175}$$

We consider the vacuum self-energy at the 3-loop level. There are

- (i) “clover-leaf graphs” from the P^2Q^4 and Q^6 couplings with three loops all meeting at one point,
- (ii) a string of three loops with two four-point vertices. We call such a string a PP string (or PQ string or QQ string) if the equal-time contractions at the ends of the string consist of two P lines (or one P and one Q line, or two Q lines. Since an equal-time PQ contraction vanishes, this exhausts all possibilities.)
- (iii) watermelon graphs with 4 propagators between 2 vertices.
- (iv) a one-loop contribution from the extra two-loop interaction ΔV ; since ΔV is of order \hbar^2 , this one-loop graph contributes also at order \hbar^3 . To evaluate $\Delta V = \frac{1}{8}\Gamma_{jk}^i\Gamma_{\ell l}^j g^{kl}$ we note that $g_{ij} = (1 + Q^2)^2$ and find

$$\Delta V = \frac{\hbar^2}{2}Q^2(1 + Q^2)^{-3} = \frac{\hbar^2}{2}Q^2 + \dots \tag{176}$$

There is no R term since the metric is flat in this model. For the phase-space calculation we use the propagators

$$\begin{aligned}
\langle P(\sigma)P(\tau) \rangle &= \langle Q(\sigma)Q(\tau) \rangle = \frac{1}{2}e^{-i|\sigma-\tau|} \\
\langle P(\sigma)Q(\tau) \rangle &= -\langle Q(\sigma)P(\tau) \rangle = \frac{-i}{2}e^{-i|\sigma-\tau|}\epsilon(\sigma - \tau)
\end{aligned} \tag{177}$$

and further the equal-time contractions

$$\langle P(\sigma)P(\sigma) \rangle = \langle Q(\sigma)Q(\sigma) \rangle = \frac{1}{2}; \quad \langle P(\sigma)Q(\sigma) \rangle = \langle Q(\sigma)P(\sigma) \rangle = 0 \tag{178}$$

We find the following results

$\frac{\text{From } P^2Q^2 - \frac{3}{2}P^2Q^4}{\text{clover} = -\frac{9i}{16}}$	$\frac{\text{From } -\frac{1}{3}Q^4 - \frac{1}{18}Q^6}{\text{clover} = -\frac{5i}{48}}$	$\underline{\text{Cross terms}}$
$P - P \text{ string} = \frac{i}{16}$	$\text{string} = \frac{i}{4}$	$QP \text{ string} = \frac{-i}{4}$
$Q - Q \text{ string} = \frac{i}{16}$		$QQ \text{ string} = \frac{i}{4}$
$P - Q \text{ string} = \frac{-i}{8}$		
$\text{watermelon (no } PQ) = \frac{i}{16}$	$\text{watermelon} = \frac{i}{24}$	$\text{watermelon (two } PQ) = \frac{i}{4}$
$\text{watermelon (two } PQ) = \frac{i}{4}$		
$\text{watermelon (four } PQ) = \frac{i}{16}$		$\underline{\text{From } \Delta V} : \frac{-i}{4}$

(179)

Adding all contributions, we find the correct result: the sum of the vacuum self-energies vanishes at the three-loop level. This demonstrates that at the 3-loop level no further counterterms are present in the phase space path integral.

In the configuration space path integral, there are extra vertices and an extra term in the $\dot{Q}\dot{Q}$ propagator. According to Matthews' theorem, the final answer should be the same as in the phase space approach. We now check this. The QQ propagator is the same, while the $\dot{Q}Q$ propagator in configuration space is equal to the PQ propagator in phase space. The $\dot{Q}\dot{Q}$ propagator differs from the PP propagator by a Dirac delta function

$$\langle \dot{Q}(\sigma)\dot{Q}(\tau) \rangle = \frac{1}{2}e^{-i|\sigma-\tau|} + i\delta(\sigma - \tau), \quad (180)$$

which is due to differentiation of the time-ordering $\theta(\sigma - \tau)$ and $\theta(\tau - \sigma)$ functions. The action contains also ghosts,

$$L(Q, \dot{Q}, b, c, a) = \frac{1}{2}(\dot{Q}\dot{Q} + bc + aa)(1 + Q^2)^2 - \frac{1}{2}Q^2(1 + \frac{1}{3}Q^2)^2, \quad (181)$$

whose only role is to cancel products of Dirac delta functions. We shall now show that the contribution from the extra vertices in the Hamiltonian approach equals the contributions of the extra terms with one Dirac delta function in the configuration space approach.

The extra vertices in the phase space approach are $L_{\text{int}}(P, Q) - L_{\text{int}}(\dot{Q}, Q)|_{\dot{Q}=P} = -2P^2Q^4$. Hence we get only an extra contribution to the clover leaf graph. Since $-3/2P^2Q^4$ gave $-9i/16$ according to (179), we now get 4/3 times this contribution

$$\text{contribution extra vertices in phase space} = -\frac{3i}{4}. \quad (182)$$

In the configuration space approach, all delta functions in the clover leaf graph cancel, since for every $\dot{Q}\dot{Q}$ contraction there is a compensating bc and aa contraction. In the graph with a string of three loops, no delta functions remain if one of the outer loops is a $\dot{Q}\dot{Q}$ contraction. The graph with one \dot{Q} in an outer loop vanish. Hence only the graph in which the inner loop contains two $\dot{Q}\dot{Q}$ contractions yields an extra contribution. This contribution comes from the graph with two $\dot{Q}^2 Q^2$ vertices. It contains a term proportional to $\delta(\sigma - \tau)^2$ which cancels against the contribution from similar graphs with an internal ghost loop. Only the terms with one delta function remain and one finds

$$\text{extra contribution to string} = -\frac{i}{4}. \quad (183)$$

Finally there are the watermelon graphs. We must consider graphs with two $\dot{Q}\dot{Q}$ propagators and graphs with one $\dot{Q}\dot{Q}$ propagator. However, the latter do not contribute because they also contain a $\langle Q(\sigma)\dot{Q}(\tau) \rangle$ and a $\langle \dot{Q}(\sigma)Q(\tau) \rangle$ propagator, and the integral $\int \delta(\sigma - \tau)\epsilon(\sigma - \tau)\epsilon(\tau - \sigma)d\sigma d\tau$ vanishes according to our rules. In the graph with two $\langle \dot{Q}\dot{Q} \rangle$ propagators we again need the term proportional to one $\delta(\sigma - \tau)$.

$$\text{extra contribution watermelon graphs} = -\frac{i}{2}. \quad (184)$$

Since (182) equals the sum of (183) and (184), Matthews' theorem is verified in this model at the three-loop level. The extra potential in (176), being proportional to two Christoffel symbols, will not contribute below the three-loop level in normal co-ordinates. However, at the three-loop level it does contribute. Note that the usual co-ordinate transformation from arbitrary co-ordinates to normal co-ordinates will yield correct results in the one- and two-loop computation, but will yield an incorrect result at the three-loop level. In our model, this is very clear: in normal co-ordinates, the action (174) reverts to a free model, but the extra potential in (176) would yield a spurious contribution in the normal co-ordinates q .

A.2 Higher derivative theories

To test our rules in a higher-derivative and higher dimensional model, we consider a massive real scalar field φ with rather singular interactions [39]

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}\lambda(\partial_\mu\varphi\partial^\mu\varphi)^2 \quad (185)$$

One novelty that appears in this example is the need to introduce 'Lee-Yang' ghosts which couple to derivatives of the scalar fields, see (203).

The conjugate momentum is defined by

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)} = \dot{\varphi} + \lambda \frac{\delta \mathcal{L}_{int}}{\delta \dot{\varphi}}(\varphi, \vec{\nabla} \varphi, \dot{\varphi}) = \dot{\varphi} + \lambda \dot{\varphi} \left[(\vec{\nabla} \varphi)^2 - \dot{\varphi}^2 \right], \quad (186)$$

and the Hamiltonian density is given by

$$\begin{aligned} \mathcal{H} &= \pi \dot{\varphi} - \mathcal{L}(\varphi, \vec{\nabla} \varphi, \dot{\varphi}) = \pi f(\varphi, \nabla \varphi, \pi; \lambda) - \mathcal{L}(\varphi, \vec{\nabla} \varphi, f) \\ &= \sum_{n=0}^{\infty} \lambda^n \mathcal{H}^{(n)}(\varphi, \nabla \varphi, \pi) \end{aligned}$$

It is straightforward to find the first few terms of \mathcal{H} ,

$$\begin{aligned} \mathcal{H}^{(0)} &= \frac{1}{2}[\pi^2 + (\vec{\nabla} \varphi)^2] + \frac{1}{2}m^2 \varphi^2 \\ \mathcal{H}^{(1)} &= -\mathcal{L}_{int}(\varphi, \vec{\nabla} \varphi, \pi) = \frac{1}{4}[\pi^2 - (\vec{\nabla} \varphi)^2]^2 \\ \mathcal{H}^{(2)} &= \frac{1}{2} \left[\frac{\delta \mathcal{L}_{int}}{\delta \dot{\varphi}}(\varphi, \vec{\nabla} \varphi, \pi) \right]^2 = \frac{1}{2} \pi^2 [\pi^2 - (\vec{\nabla} \varphi)^2]^2 \end{aligned} \quad (187)$$

The terms linear in λ yield the same vertices as in the Lagrangian approach, namely $\mathcal{L}_{int}(\varphi, \vec{\nabla} \varphi, \pi)$, but with $\dot{\varphi}$ replaced by π . Extra with respect to the Lagrangian approach are the vertices in $\mathcal{H}^{(2)}$. In the interaction picture where one uses the free field equations and in-fields φ_{in} and π_{in} , one replaces π_{in} by $\dot{\varphi}_{in}$.

On an infinite t -interval the boundary conditions on the bra and ket vacuum are that they are the lowest energy states. For the Feynman propagator one then obtains

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 + m^2 - i\epsilon} e^{ik(x-y)} \quad (188)$$

For the $\varphi\pi$ propagator one finds

$$\langle 0 | T \varphi(x) \pi(y) | 0 \rangle = \langle 0 | T \varphi(x) \dot{\varphi}(y) | 0 \rangle = \frac{\partial}{\partial y^0} \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle \quad (189)$$

because $\frac{\partial}{\partial y^0}$ hitting $\theta(x^0 - y^0)$ produces a $\delta(x^0 - y^0)$ times a commutator $[\varphi(\vec{x}, t_0), \varphi(\vec{y}, t_0)]$ which vanishes. However, for the $\pi\pi$ propagator one obtains an extra term

$$\begin{aligned} \langle 0 | T \pi(x) \pi(y) | 0 \rangle &= \langle 0 | T \dot{\varphi}(x) \dot{\varphi}(y) | 0 \rangle = \\ &= \frac{\partial}{\partial x^0} \frac{\partial}{\partial y^0} \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle - \delta(x^0 - y^0) [\varphi(x), \pi(y)] \\ &= \frac{\partial}{\partial x^0} \frac{\partial}{\partial y^0} \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle - i\hbar \delta^4(x - y) \end{aligned} \quad (190)$$

More generally we have

$$\begin{aligned} \langle 0 | T \partial_\mu \varphi(x) \partial_\nu \varphi(y) | 0 \rangle &= \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle \\ &\quad - i\hbar \delta_\mu^0 \delta_\nu^0 \delta^4(x - y) \end{aligned} \quad (191)$$

In momentum space we find

$$\langle 0|T\partial_\mu\varphi(x)\partial_\nu\varphi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \left(\frac{-ik_\mu k_\nu}{k^2 + m^2 - i\epsilon} - i\hbar\delta_\mu^0\delta_\nu^0 \right) e^{ik(x-y)}. \quad (192)$$

The same phase space propagators are obtained from the path integral by inverting the kinetic matrix of the fields φ and π .

We shall now consider elastic $\varphi\varphi$ scattering. There are the extra vertices and the extra terms in the unequal-time propagator to use, but we shall also need equal-time contractions. They are given by

$$\langle 0|\partial_\mu\varphi(x)\partial_\nu\varphi(x)|0\rangle = \sum_{\vec{k}} \frac{k_\mu k_\nu}{2\omega V} = \int \frac{d_3\vec{k}}{(2\pi)^3} \frac{k_\mu k_\nu}{2\omega} \quad (193)$$

These equal-time contractions are the same in the Lagrangian approach as in the Hamiltonian approach. We shall not try to regulate these divergent expressions, but we shall only use that for $k_\mu k_\nu$ equal to $k_i k_j$ one may replace $k_i k_j$ by $\frac{1}{3}\delta_{ij}\vec{k}^2$, while for $k_\mu k_\nu$ equal to $k_i k_0$ one obtains zero. We shall denote the corresponding divergent integrals by

$$\begin{aligned} I(\vec{k}^2) &= \int \frac{d_3\vec{k}}{(2\pi)^3} \frac{\vec{k}^2}{2\omega}, & I(k_0^2) &= \int \frac{d_3k}{(2\pi)^3} \frac{k_0^2}{2\omega} \\ I(k_i k_j) &= \frac{1}{3}\delta_{ij}I(\vec{k}^2), & I(k_i k_0) &= 0 \end{aligned} \quad (194)$$

(where $k_0^2 = \omega^2$). The equal-time contractions in the Hamiltonian approach are thus equal to those in the Lagrangian approach. Our aim is to show that the extra terms with $I(\vec{k}^2)$ and $I(k_0^2)$ in the Hamiltonian approach (due to extra vertices in H_{int}) cancel algebraically with similar terms in the Lagrangian approach (due to the term $i\delta_\mu^0\delta_\nu^0\delta^4(x-y)$ in the propagator).

Recalling the vertices

$$\begin{aligned} \mathcal{H}_{\text{int}} &= \lambda\mathcal{H}^{(1)} + \lambda^2\mathcal{H}^{(2)} + \dots \\ &= \frac{\lambda}{4}(\partial_\mu\varphi\partial^\mu\varphi)^2 + \frac{1}{2}\lambda^2\dot{\varphi}^2(\partial_\mu\varphi\partial^\mu\varphi)^2 + \dots \end{aligned} \quad (195)$$

there are 3 graphs to be computed, shown in figure 1.

In (I) we find the combination $(\langle p_\mu(x)p_\nu(y) \rangle + i\delta_\mu^0\delta_\nu^0(x-y))(\langle p_\rho(x)p_\sigma(y) \rangle + i\delta_\rho^0\delta_\sigma^0(x-y))$ where we have written the Feynman propagators as a sum of phase space propagators plus contact terms. The two cross terms yield extra terms proportional to the equal-time contractions $\langle p_\mu(x)p_\nu(x) \rangle$. Since the phase space

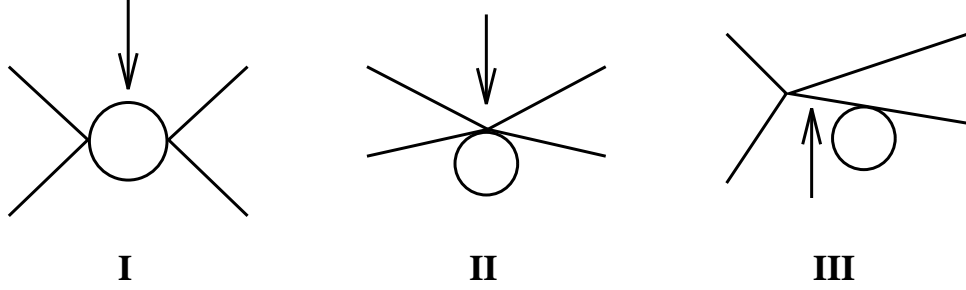


Figure 1: The three graphs to be computed. Arrows indicate where the noncovariant propagator or the extra vertex is to be used.

propagators are continuous at $x = y$, there are no subtleties in taking the equal-time limit. In (II) the equal-time contraction is the same in phase-space and in configuration space, but the vertex $\mathcal{H}_{(2)}$ is extra in phase space. Finally, in (III) the propagator in the loop is the same in phase space as in configuration space for the same reason, but the propagator connecting the two vertices has a noncovariant piece. The net effect of these noncovariant pieces in the propagator is to contract the propagator to a point. Hence, all 3 graphs become topologically of the form of figure (II), which makes the cancellation possible. We now study in more detail whether the extra contributions from (I) and (III) in the Lagrangian approach are equal to the extra contribution from (II) in the Hamiltonian approach.

For (I) we need the contractions in

$$\begin{aligned} S_I &= \frac{(-i)^2}{2!} T \int \lambda \mathcal{H}^{(1)}(x) d^4x \int \lambda \mathcal{H}^{(1)}(y) d^4(y) \\ &= \frac{-\lambda^2}{32} T \int \partial_\mu \varphi \partial^\mu \varphi \partial_\nu \varphi \partial^\nu \varphi(x) d^4x \int \partial_\rho \varphi \partial^\rho \varphi \partial_\sigma \varphi \partial^\sigma \varphi(y) d^4y \end{aligned} \quad (196)$$

As first contraction we can take $\langle \partial_\nu \varphi(x) \partial_\sigma \varphi(y) \rangle$, it comes with a statistical factor $4 \times 4 = 16$. The second contraction can then still be done in 3 different ways ($\partial_\nu \varphi \partial_\sigma \varphi$, $\partial_\nu \varphi \partial_\rho \varphi$ or $\partial_\mu \varphi \partial_\rho \varphi$); however, to avoid double-counting, we need to multiply with a factor 1/2. Hence

$$\begin{aligned} S_I &= \frac{-\lambda^2}{32} \frac{16}{2} [(\partial_\mu \varphi)^2(x) \langle \partial_\nu \varphi(x) \partial_\sigma \varphi(y) \rangle \langle \partial^\nu \varphi(x) \partial^\sigma \varphi(y) \rangle (\partial_\rho \varphi)^2(y) \\ &+ 4(\partial_\mu \varphi)^2(x) \langle \partial_\nu \varphi(x) \partial_\sigma \varphi(y) \rangle \langle \partial^\nu \varphi(x) \partial_\rho \varphi(y) \rangle \partial^\sigma \varphi(y) \partial^\rho \varphi(y) \\ &+ 4\partial_\mu \varphi(x) \partial_\nu \varphi(x) \langle \partial^\mu \varphi(x) \partial_\rho \varphi(y) \rangle \langle \partial^\nu \varphi(x) \partial_\sigma \varphi(y) \rangle \partial^\rho \varphi(y) \partial^\sigma \varphi(y)] \end{aligned} \quad (197)$$

In each pair of contractions we make the substitution

$$\begin{aligned} \langle \partial_\mu \varphi(x) \partial_\nu \varphi(y) \rangle &\langle \partial_\rho \varphi(x) \partial_\sigma \varphi(y) \rangle = \\ \langle p_\mu(x) p_\nu(y) + i\delta_\mu^0 \delta_\nu^0 \delta^4(x - y) \rangle &\langle p_\rho(x) p_\sigma(y) + i\delta_\rho^0 \delta_\sigma^0 \delta^4(x - y) \rangle \end{aligned} \quad (198)$$

and retain only the two cross terms, replacing $\delta^4(x-y)p_\rho(x)p_\sigma(y)$ by equal-time contractions for which we then use (193). This yields

$$\begin{aligned} S_I &= -\frac{i}{2}(\partial_\mu\varphi)^4 I(k_0^2) - 2i(\partial_\mu\varphi)^2\partial^\sigma\varphi\dot{\varphi}I(k_0k_\sigma) \\ &\quad - 2i\partial^\mu\varphi\partial^\rho\varphi\dot{\varphi}^2 I(k_\mu k_\rho) \end{aligned} \quad (199)$$

In S_{II} we find four different contractions, leading to

$$\begin{aligned} S_{II} &= -\frac{i}{2}\left[4\dot{\varphi}^2\partial^\mu\varphi\partial^\nu\varphi I(k_\mu k_\nu) + 2\dot{\varphi}^2(\partial\varphi)^2 I(k_\nu k^\nu)\right. \\ &\quad \left.+ 8\dot{\varphi}\partial^\mu\varphi(\partial\varphi)^2 I(k_0 k_\mu) + (\partial\varphi)^4 I(k_0, k_0)\right] \end{aligned} \quad (200)$$

Finally, in (III) there are two different contractions, yielding

$$S_{III} = -i(\partial\varphi)^2\dot{\varphi}^2 I(k_\mu k^\mu) + 2i(\partial\varphi)^2\dot{\varphi}^2 I(k_0^2) \quad (201)$$

We tabulate the results by decomposing each $(\partial_\mu\varphi)^2$ into $(\nabla\varphi)^2 - \dot{\varphi}^2$, and for notational simplicity we write \vec{k}^2 instead of $I(\vec{k}^2)$, and k_0^2 instead of $I(k_0^2)$. We find then the following extra terms

	$(\nabla\varphi)^4\vec{k}^2$	$(\nabla\varphi)^4k_0^2$	$(\nabla\varphi)^2\dot{\varphi}^2\vec{k}^2$	$(\nabla\varphi)^2\dot{\varphi}^2k_0^2$	$\dot{\varphi}^4\vec{k}^2$	$\dot{\varphi}^4k_0^2$
(I) :	0	-1/2	-2/3	1 + 2	0	-1/2 - 2 - 2
(III) :	0	0	-1	3	1	-3
(II) :	0	-1/2	-2/3 - 1	1 + 4 + 1	1	-2 - 1 - 4 - 1/2

(202)

Since the sum of (I) and (III) is equal to (II), we conclude that Matthews' theorem is satisfied in this example. Note that this conclusion is not based on a particular regularization scheme; rather the cancellation is algebraic.

Finally we must account for the terms with $[\delta^4(x-y)]^2$ in (197). In the Lagrangian approach, one integrates out the momenta by a saddle-point method. This means that one must evaluate $\frac{\partial^2}{\partial p^2}(p\dot{q} - H(p, q)) = -\frac{\partial^2}{\partial p^2}H(p, q) = -\frac{\partial^2}{\partial p^2}(p\dot{q}(p) - L(q, \dot{q}(p)))$. Using $\frac{\partial}{\partial p}L(q, \dot{q}(p)) = p\frac{\partial\dot{q}}{\partial p}$ one finds

$$\frac{\partial^2}{\partial p^2}[p\dot{q} - H(p, q)] = -\frac{\partial\dot{q}}{\partial p} = -\frac{1}{\frac{\partial p}{\partial\dot{q}}} = -\frac{1}{\frac{\partial^2 L}{\partial\dot{q}^2}}$$

Hence the a, b, c ghosts one needs to add are given by

$$L(\text{ghosts}) = b\frac{\partial^2 L}{(\partial\dot{q})^2} c + a\frac{\partial^2 L}{(\partial\dot{q})^2} a$$

In our case this yields

$$L(\text{ghosts}) = b\left(1 + \lambda\left\{(\nabla\varphi)^2 - 3\dot{\varphi}^2\right\}\right)c + \text{same with } a. \quad (203)$$

The $\delta^4(x-y)^2$ terms from (197) are easily seen to be proportional to $[(\nabla\varphi)^2 - 3\dot{\varphi}^2]^2$, and the ghost loops cancel these singular terms since the bc and aa propagators are proportional to $\delta^4(x-y)$.

This concludes our discussion of this example and of Matthews' theorem, which states that the Hamiltonian and Lagrangian approaches to perturbative quantum field theory are equivalent. We used it to test our rules in a one-dimensional and a four-dimensional model. Crucial was the correct definition of equal-time contractions. It is often stated that equal-time contractions are ill-defined, but if one wants to compute loop corrections in non-linear σ models, one must deal with them. Equal-time contractions are, in fact, already needed in a much better known area: the Ward identity for the self-energy of charged massive scalars coupled to photons are only satisfied if one includes equal-time contractions. These seagull-graphs can be computed with dimensional regularization and are nonvanishing, but since they correspond to $\langle q(x)q(x) \rangle$, they are the same in Hamiltonian and Lagrangian formalism and are the limit of x tending to y of $\langle q(x)q(y) \rangle$. The equal-time contractions we considered included $\langle \dot{q}(x)\dot{q}(x) \rangle$ and $\langle \dot{q}(x)q(x) \rangle$, and these are not the same as obtained from the limit y tending to x .

A.3 The covariant spin-3/2 Jacobian¹⁵

In order to obtain a covariant expression for the spin-3/2 transformation rule under space-time transformations, and as a consequence a covariant expression for the corresponding Jacobian, we must take certain linear combinations of Einstein transformations and local Lorentz transformations. For spin-1/2 fields, this is easy and well-known, as we now show. Afterwards we consider the more complicated case of spin-3/2 fields. To wet the appetite of the reader for this problem, we first quote the final results, which were obtained by Fujikawa for spin-1/2 [17], used by Alvarez-Gaumé and Witten [2] and further studied in [40, 41]

$$\begin{aligned}
\text{spin } 1/2 : \quad \delta_{AW} \tilde{\psi} &= \chi^\mu D_\mu \tilde{\psi} + \frac{1}{2} (D_\mu \chi^\mu) \tilde{\psi} \\
D_\mu \tilde{\psi}^a &= \partial_\mu \tilde{\psi}^a + \frac{1}{4} \omega_\mu^{mn} (\gamma_m \gamma_n)^a_b \tilde{\psi}^b - \frac{1}{2} \Gamma_{\mu\nu}{}^\nu \tilde{\psi}^a \\
\text{spin } 3/2 : \quad \delta_{AW} \tilde{\psi}_m &= \chi^\mu D_\mu \tilde{\psi}_m + \frac{1}{2} (D_\mu \chi^\mu) \tilde{\psi}_m + [(D_m \chi^n) - (D^n \chi_m)] \tilde{\psi}_n \\
D_m \chi^n &= e_m{}^\mu (\partial_\mu \chi^n + \omega_\mu{}^n{}_p \chi^p), \quad \chi^n = e_\mu{}^n \chi^\mu \\
D_\mu \tilde{\psi}_m &= \partial_\mu \tilde{\psi}_m + \omega_{\mu m}{}^n \tilde{\psi}_n + \frac{1}{4} \omega_\mu{}^{pq} \gamma_p \gamma_q \tilde{\psi}_m - \frac{1}{2} \Gamma_{\mu\nu}{}^\nu \tilde{\psi}_m \quad (204)
\end{aligned}$$

¹⁵These results were obtained in collaboration with R. Endo.

The fields $\tilde{\psi}$ and $\tilde{\psi}_m$ are world-scalar *densities* of weight 1/2, namely $\tilde{\psi} = g^{1/4}\psi$ and $\tilde{\psi}_m = g^{1/4}\psi_m$ with $\psi_m = e_m{}^\mu\psi_\mu$. Thus the covariant derivatives contain a term $\Gamma_{\mu\nu}{}^\nu$. Further the spin-3/2 field ψ_m , the so-called gravitino field, is a Lorentz vector-spinor. This explains the term $\omega_{\mu m}{}^n\tilde{\psi}_n$ in its covariant derivative.

The last two terms in the spin-3/2 transformation rule describe a local Lorentz transformation which acts on the vector indices with parameter $D_{[\mu}\chi_{n]}$, whereas the first two terms contain a local Lorentz transformation which acts on the spinor indices with parameter $\chi^\mu\omega_\mu{}^m{}_n$. At first sight these rules do not seem to correspond to a linear combination of the usual transformations. We now proceed to demystify these expressions. We begin with the easier case of spin-1/2.

For spin-1/2, we define Einstein, local Lorentz and “covariant” transformations by

$$\begin{aligned}\delta_E(\chi^\mu)\tilde{\psi} &= \chi^\mu\partial_\mu\tilde{\psi} + \frac{1}{2}(\partial_\mu\chi^\mu)\tilde{\psi} \\ \delta_{\ell L}(\lambda^{mn})\tilde{\psi} &= \frac{1}{4}\lambda^{mn}\gamma_m\gamma_n\tilde{\psi} \\ \delta_{cov}(\chi^\mu) &= \delta_E(\chi^\mu) + \delta_{\ell L}(\chi^\mu\omega_\mu{}^{mn})\end{aligned}\tag{205}$$

Thus for the spinor field $\tilde{\psi}$ one has

$$\begin{aligned}\delta_{cov}(\chi^\mu)\tilde{\psi} &= \chi^\mu\partial_\mu\tilde{\psi} + \frac{1}{2}(\partial_\mu\chi^\mu)\tilde{\psi} + \frac{1}{4}\chi^\mu\omega_\mu{}^{mn}\gamma_m\gamma_n\tilde{\psi} \\ &= \chi^\mu D_\mu\tilde{\psi} + \frac{1}{2}(D_\mu\chi^\mu)\tilde{\psi}\end{aligned}\tag{206}$$

In order to obtain a covariant result, we have added in the last line two terms with Christoffel symbols whose sum cancels, as follows from

$$\begin{aligned}D_\mu\chi^\mu &= \partial_\mu\chi^\mu + \Gamma_{\mu\nu}{}^\mu\chi^\nu \\ D_\mu\tilde{\psi} &= \partial_\mu\tilde{\psi} + \frac{1}{4}\omega_\mu{}^{mn}\gamma_m\gamma_n\tilde{\psi} - \frac{1}{2}\Gamma_{\mu\nu}{}^\nu\tilde{\psi}\end{aligned}\tag{207}$$

We can, of course, add further covariant terms. A particular combination which will play a role is

$$\begin{aligned}\delta_{sym}(\chi^\mu) &\equiv \delta_{cov}(\chi^\mu) + \delta_{\ell L}(D_{[m}\chi_{n]}) \\ D_{[m}\chi_{n]} &= \frac{1}{2}[(D_m\chi_n) - (D_n\chi_m)]\end{aligned}\tag{208}$$

For the spinor field $\tilde{\psi}$ we then obtain

$$\delta_{sym}(\chi^\mu)\tilde{\psi} = \chi^\mu D_\mu\tilde{\psi} + \frac{1}{2}(D_\mu\chi^\mu)\tilde{\psi} + \frac{1}{4}\frac{[(D^m\chi^n) - (D^n\chi^m)]}{2}\gamma_m\gamma_n\tilde{\psi}\tag{209}$$

To illustrate where this particular combination of Einstein and Lorentz transformations comes from, we evaluate δ_{sym} on the vielbein. First we compute δ_{cov} on the vielbein

$$\begin{aligned}\delta_{cov}(\chi^\nu)e_m{}^\mu &= \chi^\nu\partial_\nu e_m{}^\mu - (\partial_\nu\chi^\mu)e_m{}^\nu + \chi^\nu\omega_{\nu m}{}^n e_n{}^\mu \\ &= -\partial_m\chi^\mu + \chi^\nu(\partial_\nu e_m{}^\mu + \omega_{\nu m}{}^n e_n{}^\mu) \\ &= -\partial_m\chi^\mu + \chi^\nu(-\Gamma_{\nu\rho}{}^\mu e_m{}^\rho) \equiv -D_m\chi^\mu\end{aligned}\tag{210}$$

where we defined $\partial_m = e_m{}^\nu\partial_\nu$ and used the vielbein postulate. The transformation law defined in (208) then yields

$$\begin{aligned}\delta_{sym}(\chi^\nu)e_m{}^\mu &= -D_m\chi^\mu + \frac{1}{2}(D_m\chi^n - D^n\chi_m)e_n{}^\mu \\ &= -\frac{1}{2}(D_m\chi^\mu + D^\mu\chi_m)\end{aligned}\tag{211}$$

Hence, if the vielbein would have been symmetric in its two indices m and μ to begin with, then a transformation with δ_{sym} keeps that symmetry. For this reason one might call the transformation δ_{sym} a “symmetric Einstein transformation”, namely an Einstein transformation with parameter χ^μ which, as far as it acts on the vielbein, has been made symmetric by suitable local Lorentz transformations whose parameter also depends on χ^μ . For spin-1/2 fields, both δ_{cov} in (206) and δ_{sym} in (209) are manifestly covariant, and at this point it is not clear which one to choose for the computation of gravitational anomalies. As we shall see, δ_{AW} is actually equivalent to both. For spin-3/2, only one of them is covariant, as we now discuss.

To understand the spin-3/2 transformation rule in (204), we first compute δ_{cov} on a spin-3/2 field $\tilde{\psi}_m$. We find

$$\begin{aligned}\delta_{cov}\tilde{\psi}_m &= \chi^\mu\partial_\mu\tilde{\psi}_m + \frac{1}{2}(\partial_\mu\chi^\mu)\tilde{\psi}_m + \chi^\mu\omega_{\mu m}{}^n\tilde{\psi}_n + \frac{1}{4}\chi^\mu\omega_\mu{}^{pq}\gamma_p\gamma_q\tilde{\psi}_m \\ &= \chi^\mu D_\mu\tilde{\psi}_m + \frac{1}{2}(D_\mu\chi^\mu)\tilde{\psi}_m\end{aligned}\tag{212}$$

where again in the last line the two Christoffel symbols cancel. Hence, the proposed spin-3/2 law can be rewritten as

$$\delta_{AW}\tilde{\psi}_m = \left[\delta_{cov}(\chi^\mu) + 2\delta_{\ell L}^{(1)}(D_{[m}\chi_{n]})\right]\tilde{\psi}_m\tag{213}$$

where the superscript (1) indicates that this Lorentz transformation only acts on the spin-1 (vector) index of $\tilde{\psi}_m$, but not on its spinor index. Compare this now with the symmetric spin-1/2 rule

$$\delta_{sym}\tilde{\psi} = \left[\delta_{cov}(\chi^\mu) + \delta_{\ell L}^{(1/2)}(D_{[m}\chi_{n]})\right]\tilde{\psi}\tag{214}$$

where the superscript $(\frac{1}{2})$ indicates that this Lorentz transformation acts on the spinor indices. These rules for the spin-1/2 and spin-3/2 fields do not seem to agree at all. However, they are nevertheless equivalent, due to a surprising identity found by Fujikawa, Toniya, Yasuda and Endo [17, 41]. Namely, *if one uses $\not{D}\not{D}$ as regulator* both for the spin-1/2 case and for the spin-3/2 case, the anomalies due to $\frac{1}{2}\delta_{cov}(\chi^\mu)$ are equal to minus those coming from $\delta_{\ell L}^{(1/2)}(D_{[m}\chi_{n]})$:

$$\frac{1}{2}\delta_{cov}(\chi^\mu) \sim -\delta_{\ell L}^{(1/2)}(D_{[m}\chi_{n]}) \quad (215)$$

Using this equivalence, the proposed transformation rules for the spin-1/2 and the spin-3/2 field in (204) turn out to be the same combination of Einstein and local Lorentz transformations, after all

$$\begin{aligned} \text{spin } 1/2 : \delta_{AW} &= \delta_{cov} = 2\delta_{cov} + 2\delta_{\ell L}^{1/2}(D\chi) = 2\delta_{sym} \\ \text{spin } 3/2 : \delta_{AW} &= \delta_{cov} + 2\delta_{\ell L}^{(1)}(D\chi) \\ &= 2\delta_{cov} + 2\delta_{\ell L}^{1/2}(D\chi) + 2\delta_{\ell L}^{(1)}(D\chi) \\ &= 2\delta_{sym} \end{aligned} \quad (216)$$

Note the relative factor 2 between δ_{cov} and δ_{sym} [42].

It remains to prove that the anomalies coming from $\frac{1}{2}\delta_{cov}(\chi)$ are the same as those from $-\delta_{\ell L}^{(1)}(D\chi)$. We begin with the spin-1/2 fields. We take $\tilde{\psi}$ and $\tilde{\psi}_m$ to be left-handed, whereas the fields $\tilde{\bar{\psi}}$ and $\tilde{\bar{\psi}}_m$ are independent right-handed fields in Euclidean space. (So $\bar{\psi}$ is not ψ^\dagger in Euclidean space. In Minkowski space $\bar{\psi} = i\psi^\dagger\gamma^0$). We expand the spin-1/2 fields as follows:

$$\tilde{\psi} = \sum_n a_n \tilde{\psi}_{n,L} + \sum_\alpha c_\alpha \tilde{\chi}_{\alpha,L} \quad ; \quad \tilde{\bar{\psi}} = \sum_n b_n \tilde{\psi}_{n,R}^\dagger + \sum_\beta d_\beta \tilde{\zeta}_{\beta,R}^\dagger \quad (217)$$

where $\tilde{\psi}_n = g^{1/4}\psi_n$ and ψ_n are an orthonormal set of solutions of the Dirac equation with non-vanishing eigenvalue

$$\begin{aligned} i\not{D}\psi_n &= \lambda_n\psi_n \quad ; \quad \psi_{n,L} = \frac{1+\gamma_5}{\sqrt{2}}\psi_n \quad ; \quad \psi_{n,R} = \frac{1-\gamma_5}{\sqrt{2}}\psi_n \\ \int \tilde{\psi}_m^\dagger(x)\tilde{\psi}_n(x)dx &= \delta_{mn} \quad , \quad i\not{D}\tilde{\psi}_n = \lambda_n\tilde{\psi}_n \quad , \quad \not{D} = g^{1/4}\not{D}g^{-1/4} \end{aligned} \quad (218)$$

Of course, the chiral anomaly is equal to the number of χ 's minus the number of ζ 's, but we will rewrite this expression in a way where the difference between ψ , χ and ζ disappears.

For every eigenfunction ψ_n of the Dirac equation with nonvanishing eigenvalue λ_n , there is another one, $\gamma_5\psi_n$, with eigenvalue $-\lambda_n$. Hence, the “massive modes”

come in pairs, which can be combined on a chiral basis into $\psi_{n,L}$ and $\psi_{n,R}$, and

$$\int \tilde{\psi}_{mL}^\dagger \tilde{\psi}_{nR} dx = 0, \quad \int \tilde{\psi}_{mL}^\dagger \tilde{\psi}_{nL} dx = \int \tilde{\psi}_{mR}^\dagger \tilde{\psi}_{nR} dx = \delta_{mn} \quad (219)$$

where we used the orthogonality of ψ_m and $\gamma_5 \psi_n$. (They are orthogonal because they are eigenfunction with different eigenvalues). In addition there are some zero modes, solutions of $\not{D}\psi = 0$. These we can always take to be left- and/or right-handed. They also appear in the expansion of $\tilde{\psi}$ and $\tilde{\bar{\psi}}$, respectively, but they need not come in pairs. We have denoted them by $\chi_{\alpha,L}$ and $\zeta_{\beta,R}$ in (217).

The Jacobian J for an infinitesimal transformation $\delta\tilde{\psi}$ and $\delta\tilde{\bar{\psi}}$ follows from

$$\delta a_n = \int dx \tilde{\psi}_{n,L}^\dagger \delta\tilde{\psi}, \quad \delta b_n = \int dx \delta\tilde{\bar{\psi}} \tilde{\psi}_{n,R} \quad (220)$$

and similarly for δc_α and δd_β . For δ_{AW} we obtain then the following Jacobian

$$\begin{aligned} J_{AW} - 1 = & - \int dx \left\{ \sum_n \tilde{\psi}_{nL}^\dagger \left[\chi^\mu D_\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \tilde{\psi}_{n,L} \right. \\ & + \sum_\alpha \tilde{\chi}_{\alpha L}^\dagger \left[\chi^\mu D_\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \tilde{\chi}_{\alpha L} \\ & + \sum_n \tilde{\psi}_{n,R}^\dagger \left[\overleftarrow{D}_\mu \chi^\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \tilde{\psi}_{n,R} \\ & \left. + \sum_\beta \tilde{\zeta}_{\beta,R}^\dagger \left[\overleftarrow{D}_\mu \chi^\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \tilde{\zeta}_{\beta,R} \right\} \end{aligned} \quad (221)$$

where $\psi \overleftarrow{D}_\mu = \partial_\mu \psi^\dagger - \frac{1}{4} \psi^\dagger \omega_\mu^{mn} \gamma_m \gamma_n$. The minus sign always appears in Jacobians for fermions. We now rewrite the massive modes in terms of $\tilde{\psi}_n$ and $\gamma_5 \tilde{\psi}_n$, but the zero modes we keep as they appear, except that we add a factor γ_5 or $-\gamma_5$ to the left-handed or right-handed zero modes, respectively. (These factors γ_5 or $-\gamma_5$ equal unity, of course). This yields

$$\begin{aligned} J_{AW} - 1 = & - \int dx \left\{ \sum_n \tilde{\psi}_n^\dagger \left[\chi^\mu D_\mu + \overleftarrow{D}_\mu \chi^\mu + (D_\mu \chi^\mu) \right] \tilde{\psi}_n \right. \\ & \left. + \sum_n \tilde{\psi}_n^\dagger \gamma_5 \left[\chi^\mu D_\mu - \overleftarrow{D}_\mu \chi^\mu \right] \tilde{\psi}_n \right\} \\ & - \int dx \left\{ \sum_\alpha \tilde{\chi}_{\alpha L}^\dagger \left[\chi^\mu D_\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \gamma_5 \tilde{\chi}_{\alpha L} \right. \\ & \left. + \sum_\beta \tilde{\zeta}_{\beta R}^\dagger \left[\overleftarrow{D}_\mu \chi^\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \gamma_5 \tilde{\zeta}_{\beta R} \right\} \end{aligned} \quad (222)$$

Partially integrating, all massive modes without factor γ_5 cancel, while the rest yields

$$\begin{aligned} J_{AW} - 1 = & - \int dx \left[\sum_n \tilde{\psi}_n^\dagger \gamma_5 [2\chi^\mu D_\mu + (D_\mu \chi^\mu)] \tilde{\psi}_n \right. \\ & \left. + \sum_\alpha \tilde{\chi}_{\alpha L}^\dagger \gamma_5 \left[\chi^\mu D_\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \chi_{\alpha L} + \sum_\beta \tilde{\zeta}_{\beta R}^\dagger \gamma_5 \left[\chi^\mu D_\mu + \frac{1}{2}(D_\mu \chi^\mu) \right] \tilde{\zeta}_{\beta R} \right] \end{aligned} \quad (223)$$

We would like to recognize this as a sum over a complete set of states. At first sight there is a mismatch in the first term which seems a factor 2 too large with respect to the other terms. However, the complete set of massive solutions contains not only $\tilde{\psi}_n$ but also $\gamma_5 \tilde{\psi}_n$, and rewriting $2\tilde{\psi}_n \gamma_5 \tilde{\psi}_n$ for the massive modes as $\tilde{\psi}_n \gamma_5 \tilde{\psi}_n + \tilde{\psi}_{-n} \gamma_5 \tilde{\psi}_{-n}$ with $\tilde{\psi}_{-n} \equiv \gamma_5 \tilde{\psi}_n$ by definition, we arrive at

$$J_{AW} - 1 = - \int dx \sum_N \tilde{\varphi}_N^\dagger \gamma_5 \left[\chi^\mu D_\mu + \frac{1}{2} (D_\mu \chi^\mu) \right] \tilde{\varphi}_N \quad (224)$$

where $\tilde{\varphi}_N$ is the complete set of eigenfunctions, so that N runs over all eigenfunctions: $n > 0$, $n < 0$, and all zero modes.

To regulate the infinite sum over N , we add to each term a factor $\exp(-\lambda_N^2/M^2)$ and will let M tend to infinity at the end. Using

$$e^{-\lambda_N^2/M^2} \varphi_N = e^{\tilde{\not{D}} \not{D}/M^2} \varphi_N \quad (225)$$

we obtain

$$J_{AW} - 1 = - \int dx \sum_N \tilde{\varphi}_N^\dagger \gamma_5 \left[\chi^\mu D_\mu + \frac{1}{2} (D_\mu \chi^\mu) \right] e^{\tilde{\not{D}} \not{D}/M^2} \tilde{\varphi}_N \quad (226)$$

where we recall the definition $\tilde{\not{D}} = g^{1/4} \not{D} g^{-1/4}$, with \not{D} without $\Gamma_{\mu\nu}{}^\nu$ term. All terms with $\Gamma_{\mu\nu}{}^\nu$ cancel in (226), so from now on no explicit $\Gamma_{\mu\nu}{}^\nu$ are present.

Since the set φ_N is a complete set $\sum_N \tilde{\varphi}_N(x) \tilde{\varphi}_N^\dagger(y) = \delta(x-y)$, we have

$$\begin{aligned} J_{AW} - 1 &= -\text{Tr} \left(\gamma_5 \frac{1}{2} (\chi^\mu D_\mu + D_\mu \chi^\mu) e^{\tilde{\not{D}} \not{D}/M^2} \right) \\ D_\mu &= \partial_\mu + \frac{1}{4} \omega_\mu{}^{mn} \gamma_m \gamma_n \end{aligned} \quad (227)$$

where the trace Tr indicates integrating over space and summing over spinor indices.

This trace has been evaluated using plane waves. Due to the $g^{\pm 1/4}$ in $\tilde{\not{D}}$, the Einstein anomaly indeed did cancel [37], see below (103). However, by using the cyclicity of the trace we can rewrite this as

$$J_{AW} - 1 = -\text{Tr} \left(\frac{1}{2} \gamma_5 (\chi^\mu g^{-1/4} D_\mu g^{1/4} + g^{-1/4} D_\mu g^{1/4} \chi^\mu) e^{\not{D} \not{D}/M^2} \right) \quad (228)$$

In the corresponding non-linear sigma model this becomes

$$J_{AW} - 1 = \frac{1}{2\pi\hbar} \text{Tr} (\gamma_5 (\chi^i \pi_i + \pi_i \chi^i) \exp(-\beta \hat{H}/\hbar)) \quad (229)$$

with \hat{H} given in (113) and $\pi_i = p_i - \frac{\hbar i}{2} \omega_{iab} \psi^a \psi^b$. Note that the Jacobian is Weyl-ordered. One can from here compute the gravitational anomalies.

For local Lorentz transformations,

$$\delta\tilde{\psi} = \frac{1}{4}\lambda^{mn}\gamma_m\gamma_n\tilde{\psi}, \quad \delta\tilde{\bar{\psi}} = -\frac{1}{4}\lambda^{mn}\tilde{\bar{\psi}}\gamma_m\gamma_n \quad (230)$$

one may proceed in a similar manner. One finds then for chiral spinors $\tilde{\psi}$ the following Lorentz anomaly

$$J_{\ell L} - 1 = -\text{Tr} \left(\gamma_5 \left[\frac{1}{4}\lambda^{mn}\gamma_m\gamma_n \right] e^{\tilde{\not{D}}\tilde{\not{D}}/M^2} \right) \quad (231)$$

Using cyclicity of the trace, all factors of $g^{1/4}$ and $g^{-1/4}$ now cancel, and one obtains

$$J_{\ell L} - 1 = -\text{Tr} \left(\frac{1}{4}\gamma_5\lambda^{mn}\gamma_m\gamma_n \exp(-\beta\hat{H}/\hbar) \right) \quad (232)$$

Now that we have found expressions for the Einstein and Lorentz anomalies, we can prove the relation $\delta_{cov} \sim -\frac{1}{2}\delta_{\ell L}$ for the spin-1/2 fields. We use the identity

$$\int \text{Tr} \left(\gamma_5 \left(\not{\chi}\not{D} + \not{D}\not{\chi} \right) e^{\tilde{\not{D}}\tilde{\not{D}}/M^2} \right) = 0 \quad (233)$$

which follows from cyclicity of the trace and $\not{D}\gamma_5 = -\gamma_5\not{D}$. Since the difference between $\not{D}\not{D}$ as regulator and $\tilde{\not{D}}\tilde{\not{D}}$ involves terms which are proportional to M^{-2} in the path integral, and only the M -independent terms yield the anomaly while there are no singular terms in M for chiral anomalies, we may replace $\tilde{\not{D}}\tilde{\not{D}}$ by $\not{D}\not{D}$ in (233). Then we can indeed use the cyclicity of the trace. (Chiral anomalies are rather insensitive to the details of the regulator. For trace anomalies, such details do matter, but for trace anomalies fortunately we do not need (233)). Hence

$$\begin{aligned} 0 &= \int \text{Tr} \left(\gamma_5 \left(\chi^\mu D_\mu + D_\mu \chi^\mu + \gamma^{[\mu}\gamma^{\nu]} (\chi_\mu D_\nu + D_\mu \chi_\nu) \right) e^{\not{D}\not{D}/M^2} \right) dx \\ &= \int \text{Tr} \left(\left[\gamma_5 (\chi^\mu D_\mu + D_\mu \chi^\mu) + \gamma_5 \gamma^{[\mu}\gamma^{\nu]} (\chi_\mu D_\nu - D_\nu \chi_\mu) \right] e^{\not{D}\not{D}/M^2} \right) dx \\ &= \int \text{Tr} \left(\left[\gamma_5 (\chi^\mu D_\mu + D_\mu \chi^\mu) + \gamma_5 \gamma^{[\mu}\gamma^{\nu]} (D_\mu \chi_\nu) \right] e^{\not{D}\not{D}/M^2} \right) dx \\ &= -2\mathcal{A}_{n,cov}(\chi^\mu) - 4\mathcal{A}_{n,\ell L}(D_{[m}\chi_{n]}) \end{aligned} \quad (234)$$

This concludes the proof of the identity in (215) for spin-1/2.

For spin-3/2 we use the same identity in (233), and all steps in (234) are the same. Note that the local Lorentz transformations with $D_{[m}\chi_{n]}$ only act on the spin-1/2 indices of the spin-3/2 field. Thus (215) is also proven for spin-3/2. The regulator $\tilde{\not{D}}\tilde{\not{D}}$ now acts in a combined spinor-vector space, but it is diagonal in the vector indices, i.e., it acts on ψ_m the same for all m .

To see whether this regulator is obtained from the gauge-fixed spin-3/2 action, we note that the spin-3/2 action in d dimension reads

$$\mathcal{L}_0 = -\frac{1}{2}\bar{\psi}_\mu \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} D_\nu \psi_\rho \quad (235)$$

After adding a gauge fixing term [43]

$$\mathcal{L}_{\text{fix}} = \frac{d-2}{8} \bar{\psi}_\mu \gamma^\mu \gamma^\nu \gamma^\rho D_\nu \psi_\rho \quad (236)$$

and choosing a new basis for the spin-3/2 fields

$$\chi_\mu = \psi_\mu - \frac{1}{2} \gamma_\mu \gamma \cdot \psi \quad (237)$$

the action becomes a sum of Dirac actions [42]

$$\mathcal{L}_0 + \mathcal{L}_{\text{fix}} = \bar{\chi}_\mu \not{D} \chi^\mu = \bar{\chi}_m \not{D} \chi^m \quad (238)$$

The field χ_m transforms of course in the same way as ψ_m under space-time transformations and the regulator for the spin-3/2 field χ_m is thus $\not{D} \not{D}$, for the same reasons as for the spin-1/2 field.

A.4 Chiral anomalies in gravitational couplings from Feynman diagrams in quantum mechanics

In this appendix we will derive the chiral anomaly in gravitational couplings using Feynman diagrams and the Feynman rules developed in the text. Starting point will be equations (126) and (129) derived in section 3.1. As explained there, we only have to consider one-loop diagrams. An arbitrary connected one-loop diagram consists of k vertices from (129) and k propagators. There are in principal many different ways to contract the vertices with each other, but using partial integration (in this case one can verify it is allowed) and the anti-symmetry of R_{ijab} in the indices i, j , one can always move the time derivative in (129) from q^j to q^i . This implies that all different contractions of the vertices yield the same contribution. The total number of contractions is $2^{k-1}(k-1)!$, which combines with the factor $1/k!$ from expanding (129) in the total symmetry factor $2^{k-1}/k$ for each diagram. If we always contract a \dot{q} from one vertex with a q from another vertex, we find the following expression for S_{loops} , the set of connected one-loop diagrams

$$-\frac{1}{\hbar} S_{\text{loops}} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{k} \left(\frac{-R}{4\beta} \right)^k (-\beta \hbar)^k a_k \quad (239)$$

where R^k is $R_{i_1 i_2}^{i_2} R_{i_2 i_3}^{i_3} \dots R_{i_k i_1}^{i_1}$ with $R_j^i = g^{ik}(x_0) R_{kjab}(\omega(x_0)) \psi_{\text{bg}}^a \psi_{\text{bg}}^b$, and a_k denotes the integral

$$a_k = \int_{-1}^0 d\tau_1 \dots \int_{-1}^0 d\tau_k (\tau_2 + \theta(\tau_1 - \tau_2)) \dots (\tau_k + \theta(\tau_{k-1} - \tau_k)) (\tau_1 + \theta(\tau_k - \tau_1)) \quad (240)$$

This expression is cyclically symmetric. To proceed, we compute $\sum_{k=1}^{\infty} \frac{y^k}{k} a_k$. This can be rewritten as

$$\begin{aligned} \frac{y}{2} + \int_0^1 \frac{d\alpha}{\alpha} \left(\alpha \frac{d}{d\alpha} \right) & \left(\sum_{k=1}^{\infty} \frac{y^k}{k} \int_{-1}^0 d\tau_1 \dots \int_{-1}^0 d\tau_k \right. \\ & \left. \times (\alpha\tau_2 + \theta(\tau_1 - \tau_2)) \dots (\alpha\tau_k + \theta(\tau_{k-1} - \tau_k)) (\alpha\tau_1 + \theta(\tau_k - \tau_1)) \right). \end{aligned} \quad (241)$$

Here, we introduced a parameter α in the expression for a_k , which will turn out to make things much simpler. Notice that if we put this parameter equal to zero, we are left with an integral over a product of theta functions, which vanishes, except when $k = 1$, and then it equals $1/2$ in view of $\theta(0) = 1/2$, and this explains the first term $y/2$ in (241). The derivative $d/d\alpha$ in (241) can act on any of the k factors in the integral, but by cyclic symmetry these all give the same contribution and we choose it to hit only the last factor and put an extra factor of k in front. This yields

$$\frac{y}{2} + \int_0^1 \frac{d\alpha}{\alpha} \left(\sum_{k=1}^{\infty} y^k \int_{-1}^0 d\tau_1 \dots \int_{-1}^0 d\tau_k (\alpha\tau_1) (\alpha\tau_2 + \theta(\tau_1 - \tau_2)) \dots (\alpha\tau_k + \theta(\tau_{k-1} - \tau_k)) \right). \quad (242)$$

Note that we now have broken the cyclic symmetry of (240). If we expand the brackets inside the integral, we get a large sum of terms, each of which factors into a product of integrals of the form

$$\int_{-1}^0 d\tau_m \dots \int_{-1}^0 d\tau_{m+l} (\alpha\tau_m) \theta(\tau_m - \tau_{m+1}) \dots \theta(\tau_{m+l-1} - \tau_{m+l}). \quad (243)$$

For $l = 0$, the integrand is just $\alpha\tau_m$. This integral depends only on l , enabling us to write (242) as

$$\frac{y}{2} + \int_0^1 \frac{d\alpha}{\alpha} \left(\sum_{p=1}^{\infty} \left[\sum_{k=1}^{\infty} y^k \int_{-1}^0 d\tau_1 \dots \int_{-1}^0 d\tau_k (\alpha\tau_1) \theta(\tau_1 - \tau_2) \dots \theta(\tau_{k-1} - \tau_k) \right]^p \right). \quad (244)$$

The integral in (243) equals $-\alpha/(l+2)!$, and we find

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{y^k}{k} a_k &= \frac{y}{2} + \int_0^1 \frac{d\alpha}{\alpha} \left(\sum_{p=1}^{\infty} \left[\sum_{k=1}^{\infty} y^k \frac{-\alpha}{(k+1)!} \right]^p \right) \\ &= \frac{y}{2} + \int_0^1 \frac{d\alpha}{\alpha} \left(\sum_{p=1}^{\infty} \left[\frac{-\alpha}{y} (e^y - 1 - y) \right]^p \right) \\ &= \frac{y}{2} + \int_0^1 \frac{d\alpha}{\alpha} \left(\frac{1}{1 + \frac{\alpha}{y} (e^y - 1 - y)} - 1 \right) \\ &= \log \left(\frac{y/2}{\sinh(y/2)} \right) \end{aligned} \quad (245)$$

This shows that

$$-\frac{1}{\hbar} S_{\text{loops}} = \frac{1}{2} \log \left(\frac{\hbar R/4}{\sinh(\hbar R/4)} \right) \quad (246)$$

Thus, after a rescaling of the fermions, we find that the anomaly is given by

$$\mathcal{A}_n = \int \prod_{i=1}^n dx_0^i \sqrt{g(x_0)} \int \prod_{a=1}^n d\psi_{\text{bg}}^a \exp \left[\frac{1}{2} \log \left(\frac{-iR/8\pi}{\sinh(-iR/8\pi)} \right) \right] \quad (247)$$

Clearly, there is only a gravitational γ_5 anomaly in $n = 4k$ dimensions. In principle, this expression is ambiguous, since it is not clear whether R^4 means $\text{tr}(R^2)\text{tr}(R^2)$ or $\text{tr}(R^4)$. However, the precise meaning follows from the derivation above. First one has to write down a Taylor series for the logarithm, then one has to replace R^m by $\text{tr}(R^m)$ everywhere, and only then should one take the exponential. The advantage of writing the anomaly in the form (247) is that it is given directly in terms of R , rather than in terms of its eigenvalues [2].

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